MODEL THEORY: ASSIGNMENT 1

Due: Thursday, 17th August, after the lecture

Notational conventions:

- Quantifiers have *narrow scope*, i.e., they apply to the shortest possible formula following the quantifier. For example, $\forall x \varphi(x) \to \phi(x)$ is equivalent to $(\forall x \varphi(x)) \to \phi(x)$ and **not to** $\forall x(\varphi(x) \to \phi(x))$. Notation $\forall x \colon \varphi$ can be used to reduce clutter, and then the colon gives the quantifier *wide scope*. For example, $\forall x \colon \varphi(x) \to \phi(x)$ is equivalent to $\forall x(\varphi(x) \to \phi(x))$.
- $\vee$ and $\wedge$ bind stronger than $\to$ and $\leftrightarrow$. For example $\alpha \wedge \beta \to \gamma \vee \delta$ is equivalent to $(\alpha \wedge \beta) \to (\gamma \vee \delta)$ and **not to** either of $(\alpha \wedge (\beta \to \gamma)) \vee \delta$, $\alpha \wedge ((\beta \to \gamma) \vee \delta)$.
- $\vee$ and $\wedge$ have equal binding strength. For example, $\alpha \wedge \beta \vee \gamma$ is not well-formed. It is ambiguous between $\alpha \wedge (\beta \vee \gamma)$ and $(\alpha \wedge \beta) \vee \gamma$. But $\alpha \wedge \beta \wedge \gamma$ is allowed
- $\to$ and $\leftrightarrow$ have equal binding strength. For example, $\alpha \to \beta \to \gamma$ is not well-formed. It is ambiguous between $(\alpha \to \beta) \to \gamma$ and $\alpha \to (\beta \to \gamma)$.
- Binary relations can be written in infix notation. For example, $R(x, y)$ can be written as xRy. The relations usually written in infix notation, such as $\leq$, will be so written.

Question 1. *Recall that in our "official" language, conjunction, disjunction, equivalence and the existential quantifier are abbreviations:*

- $\alpha \wedge \beta$ *is* $\neg(\alpha \to \neg\beta)$
- $\alpha \vee \beta$ *is* $\neg\alpha \to \beta$
- $\alpha \leftrightarrow \beta$ *is* $(\alpha \to \beta) \wedge (\beta \to \alpha)$
- $\exists x \varphi$ *is* $\neg\forall x \neg\varphi$

We add two more abbreviations:

- $\alpha \mid \beta$ *is* $\neg\alpha \vee \neg\beta$
- $\alpha \veebar \beta$ *is* $(\alpha \vee \beta) \wedge \neg(\alpha \wedge \beta)$

Write the definition of satisfaction for each of these explicitly (as if they were not abbreviations, but basic connectives). For example, for $\wedge$ we have:

- $\mathcal{S} \models_v \alpha \wedge \beta$ *if* $\mathcal{S} \models_v \alpha$ *and* $\mathcal{S} \models_v \beta$

Continue in this fashion. You do not need to prove your answers are correct: if they are it will be easy to see.

Question 2. *Let T be the following theory:*

(1) $\forall x \exists y \colon xRy$
(2) $\forall x \forall y \colon xRy \to \neg(yRx)$
(3) $\forall x \forall y \exists z \colon xRy \to xRz \wedge zRy$
(4) $\forall x \forall y \forall z \colon xRy \wedge yRz \to xRz$
(5) $\forall x \forall y \colon \neg(x = y) \to xRy \vee yRx$

Find a structure $\mathcal{S} = (S, R^S)$ such that $\mathcal{S} \models T$. (Hint: it must be infinite.) Do not give a detailed proof that your answer is correct, but give some justification.

Question 3. *Let T be the theory from Question 2. For each $i \in \{1,2,3,4,5\}$ let φ_i be the sentence numbered (i) in Question 2. For each i above find a structure $\mathcal{M}_i$ such that $\mathcal{M}_i \models \varphi_j$ for $j \neq i$, but $\mathcal{M}_i \not\models \varphi_i$.*

Question 4. *Let $\mathcal{S}$ be a pure equality structure (no relations other than equality, no functions, no constants). For each n find a sentence φ_n with the following property: $\mathcal{S} \models \varphi_n$ iff S has at most n elements.*

Question 5. *Recall that a theory T is* complete *if for any sentence φ (in the language $\mathcal{L}$ of T), we have $\varphi \in T$ or $\neg\varphi \in T$. Now, let $\mathcal{M}$ be an $\mathcal{L}$-structure, and let $T = \{\varphi\colon \varphi$ is a sentence, and $\mathcal{M} \models \varphi\}$. Prove that T is a complete theory.*

Question 6. *Can one embed:*

1. $(\mathbb{N}_0, \leq, +, 0)$ *into* $(\mathbb{N}_0, \leq, \cdot, 1)$*?*
2. $(\mathbb{N}_0, \leq, \cdot, 1)$ *into* $(\mathbb{N}_0, \leq, +, 0)$*?*
3. $(\mathbb{N}, \cdot, 1)$ *into* $(\mathbb{N}_0, +, 0)$*?*

If an embedding exists, just state what it is; if not, prove that none exists.

In the next four questions, we explore the notion of an elementary substructure (written $\prec$) in comparison to the usual substructure (written $\leq$). We assume that $\mathcal{A} \prec \mathcal{B}$ always implies $A \subseteq B$. (The same symbol ($\prec$) is also used for the relation of "being elementarily embeddable into", which does not have that implication.)

Question 7. *Prove that the relation $\prec$ of "being an elementary substructure of" is an ordering relation. That is, for arbitrary structures $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ of the same signature the following hold:*

1. $\mathcal{A} \prec \mathcal{A}$
2. $\mathcal{A} \prec \mathcal{B}$ *and* $\mathcal{B} \prec \mathcal{A}$ *implies* $\mathcal{A} = \mathcal{B}$
3. $\mathcal{A} \prec \mathcal{B}$ *and* $\mathcal{B} \prec \mathcal{C}$ *implies* $\mathcal{A} \prec \mathcal{C}$

Question 8. *Prove that if $\mathcal{A} \leq \mathcal{B}$, $\mathcal{B} \leq \mathcal{C}$, $\mathcal{A} \prec \mathcal{C}$ and $\mathcal{B} \prec \mathcal{C}$, then $\mathcal{A} \prec \mathcal{B}$.*

Question 9. *Give a counterexample showing that the condition $\mathcal{B} \prec \mathcal{C}$ is necessary.*

Question 10. *Give a counterexample showing that the condition $\mathcal{A} \prec \mathcal{C}$ is necessary.*

The last three questions deal with diagrams.

Question 11. *Let K_2 be the complete graph on two vertices. Write the diagram of K_2 (use a, b for the new constants and E for the edge relation).*

Question 12. *(A) Let $T = (\{e\}; \cdot, {}^{-1}, e)$ be the trivial group. Show that the positive diagram of T is infinite (you do not need any new constants).*

(B) Let $\mathcal{A}$ be any structure. Show that if the signature of $\mathcal{A}$ contains a function, then the positive diagram of $\mathcal{A}$ is an infinite set of sentences.

Question 13 (Ex.3, p.17 from Shorter Model Th.)**.** *Let $(A, \overline{a})$ and $(B, \overline{b})$ be $L(\overline{c})$-structures (that is, structures in the language expanded by a sequence $\overline{c}$ of new constants, interpreted as elements of $\overline{a}$ in $(A, \overline{a})$ and as elements of $\overline{b}$ in $(B, \overline{b})$). Assume that A and B are generated by $\overline{a}$ and $\overline{b}$, respectively, and that $(A, \overline{a})$ and $(B, \overline{b})$ satisfy exactly the same atomic sentences of $L(\overline{c})$. Show that there is an isomorphism $f\colon A \to B$, such that the image of $\overline{a}$ under f is $\overline{b}$.*