MODEL THEORY: ASSIGNMENT 2
AROUND ULTRAPRODUCTS
DUE: WEEK 12

Question 1. *Consider a signature consisting of countably many binary relations. The usual composition of binary relations can be defined in this type by the following formula schema:*

$$\forall x, y\colon (R \circ S)(x, y) \leftrightarrow \exists z\colon R(x, z) \wedge S(z, y)$$

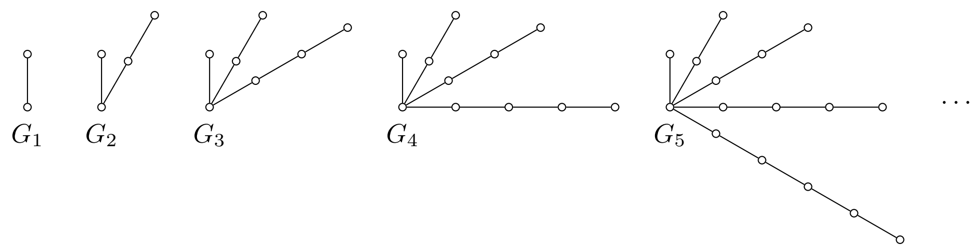
where $R \circ S$ is a new relation symbol (viewed as a single "letter"), and R, S rangle over all relations in the type. We can then view $(R \circ S)(x, y)$ as a shorthand for $\exists z\colon R(x, z) \wedge S(z, y)$, and we can extend this notation further, e.g., writing $((R \circ S) \circ T)(x, y)$. Associativity of composition of binary relations can be written in our shorthand as

$$\forall x, y\colon ((R \circ S) \circ T)(x, y) \leftrightarrow (R \circ (S \circ T))(x, y)$$

Expand the above to a formula that does not use the shorthand (i.e., to a formula with no occurrences of $\circ$), and write that formula in prenex disjunctive normal form.

Question 2. *Let $(\mathcal{A}_i)_{i \in I}$ be an infinite family of structures of the same type, such that for a cofinite set $J \subseteq I$ the structures $\mathcal{A}_j$ for all $j \in J$ are isomorphic to some finite structure $\mathcal{B}$. Let U be an ultrafilter over I. Prove that if U is not principal, then $\prod_{i \in I} \mathcal{A}_i / U$ is isomorphic to $\mathcal{B}$.*

Question 3. *Let $(\mathcal{G}_n)_{n \in \mathbb{N}}$ be the following family of graphs:*



viewed as structures in the signature of one binary relation, say E, interpreted as the edge relation. Let U be a non-principal ultrafilter on $\mathbb{N}$. Let $G = \prod_{n \in \mathbb{N}} G_n / U$. Show:

1. *That there is an element $g \in G$ with infinitely many neighbours.*
2. *That G is not connected (i.e., there are two vertices not connected by any finite path).*
3. *That G has a connected component isomorphic to the graph Z below*

4. *That $(\mathcal{G}_n)_{n \in \mathbb{N}}$ is not an elementary class (i.e., it is not axiomatisable by any set of first-order sentences).*

Question 4. *Let $\mathbb{N}_0$ be the usual structure of natural numbers (including 0) in the usual signature with $\leq$, $+$, $\times$, 0 and 1. Let U be a nonprincipal ultrafilter on ω, and let $\mathbf{N}$ be the ultrapower $\mathbb{N}_0^\omega / U$. We will identify $\mathbb{N}_0$ with its image in $\mathbb{N}_0^\omega / U$ under the natural (diagonal) embedding, and refer to the elements of $\mathbb{N}_0$ as* standard *natural numbers. The definition of a prime number remains standard: x is prime if $\forall y, z\colon x = y \times z \Rightarrow (y = 1 \vee z = 1)$. Prove:*

1. *That in $\mathbf{N}$ there exist non-standard primes: primes that are greater than any standard prime.*
2. *That any element of $\mathbf{N}$ is smaller than some (non-standard) prime.*

Question 5. *Let $\mathbb{N}_0$ and $\mathbf{N}$ be as in the previous question.*

1. *Exhibit an element of $\mathbf{N}$ which is divisible by every standard prime. To be precise: $a \in \mathbf{N}$ such that for every prime $p \in \mathbb{N}_0$ ther exists $b \in \mathbf{N}$ such that $a = b \times p$.*
2. *Prove that not every element of $\mathbf{N}$ is a product of primes. To be precise: prove that there exists $u \in \mathbf{N}$ such that $u = p_1 \times p_2 \times \cdots \times p_k$ does not hold for any* finite *set $\{p_1, p_2, \ldots, p_k\}$ of (standard or non-standard) primes.*