MODEL THEORY: ASSIGNMENT 3
AROUND GAMES AND FRAÏSSÉ LIMITS
DUE: WEEK 12

Question 1. *Let $\mathbb{K}$ be the class of strict dense linear orders. Define Φ to be the following set of formulae:*

$$\big\{\exists x \forall y\colon x = y \vee x < y,\ \exists x \forall y\colon x = y \vee y < x,\ \forall y\colon x = y \vee x < y,\ \forall y\colon x = y \vee y < x,\ x < y\colon x, y \text{ are variables}\big\}$$

Prove that Φ is an elimination set for $\mathbb{K}$.

Question 2. *Consider two countably infinite structures $\mathcal{A}$ and $\mathcal{B}$ in the language of a single binary relation R. Assume that the relation R is an equivalence relation on both $\mathcal{A}$ and $\mathcal{B}$, with:*

- *$\mathcal{A}$ contains one block of size n for each natural number n, and no other blocks aside from these;*
- *$\mathcal{B}$ contains one block of size n for each natural number n but also one infinite block.*

(i) *Show that $\mathcal{A}$ and $\mathcal{B}$ are* not *back and forth equivalent, by giving a strategy for $\forall$belard to play that will defeat $\exists$loise in the ω back and forth game.*
(ii) *Show that for any $k \in \mathbb{N}$, $\exists$loise does have a strategy for winning the k-round back and forth game. Thus $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent.*
(iii) *Give an example of an embedding of $\mathcal{A}$ into $\mathcal{B}$ that is not an elementary embedding.*
(iv) *Let T be the theory of $\mathcal{A}$. Show that T does* not *admit quantifier elimination, by showing that T is* not *model complete.*

Question 3. *Consider two countably infinite structures $\mathcal{C}$ and $\mathcal{D}$ in the language of a single binary relation R. Assume that the relation R is an equivalence relation on both $\mathcal{C}$ and $\mathcal{D}$, with:*

- *$\mathcal{C}$ contains n blocks of size n for each natural number n, and no other blocks aside from these;*
- *$\mathcal{D}$ contains 1 block of size n for each natural number n, and no other blocks aside from these.*

(i) *Find a number k such that $\forall$belard can always win the k-round back and forth game.*
(ii) *What is the smallest value for k for which part (i) is true? Prove this by sketching a strategy for $\exists$loise to win the $(k-1)$-round game.*
(iii) *Give an example of a sentence holding on $\mathcal{C}$ but not on $\mathcal{D}$. Try to minimise the quantifier rank.*

Question 4. *Let L be the language consisting of two unary relation symbols R_1 and R_2. For $i \in \{0, 1\}$ let $\mathbb{K}_i$ be the class of all L-structures defined as follows: if $\mathcal{A} \in \mathbb{K}_i$, then*

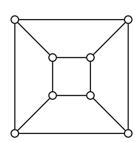
- $R_i^{\mathcal{A}} = A$
- $R_{1-i}^{\mathcal{A}} = \varnothing$.

Define $\mathbb{K} = \mathbb{K}_1 \cup \mathbb{K}_2$. Show that $\mathbb{K}$ has the amalgamation property but does not have the joint embedding property.

Question 5. *The countable complete graph K_ω is defined to be the structure $(A; E)$, where the underlying set of vertices, A, has cardinality ω, and E is a binary relation such that $(v_1, v_2) \in E$, for every pair of* distinct *vertices v_1 and v_2.*

(1) *Describe the age of K_ω.*
(2) *Prove that the age of K_ω is an amalgamation class.*
(3) *Prove that K_ω is homogeneous (use Fraïssé's Theorem).*

Question 6. *Consider the graph $\mathcal{G}$:*



Show that $\mathcal{G}$ is not homogeneous.

Question 7. *Let L be the signature consisting of a single binary relation E. Let $\mathcal{A}$ be a countable structure such that $E^{\mathcal{A}}$ is an equivalence relation on the underlying set A of $\mathcal{A}$ with the following properties:*

- *E has infinitely many blocks (that is, equivalence classes),*
- *each E block has exactly 2 elements.*

Using Fraïssé's Theorem, prove that the Fraïssé limit of the age of $\mathcal{A}$ is isomorphic to $\mathcal{A}$.