Who and what

- Lecturer:
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  Come and see me any time any day except Friday.
- Course structure:
  - One two-hour lecture a week: Thursday 10am–12noon.
  - Three assignments, each worth 20%, due in weeks 4, 8, 12.
  - Take home exam: two sections, A and B, each worth 20%.
  - Final mark computed as:
    Assgn 1 + Assgn 2 + Assgn 3 + Sect A + Sect B.
  - Section B will be noticeably more difficult than Section A.
  - In practice, this means that you will be able to get up to 80% based on the assignments and Section A alone. For more than 80% you will need Section B.

Course materials

- Website:
  `http://tomasz-kowalski.ltumathstats.com/model-theory`
  Some quite incomplete course notes are there already. Slides from each lecture will be there, and in due course, assignments.
- Main textbooks:
  Wilfrid Hodges, *Shorter Model Theory.*
  David Marker, *Model Theory. An Introduction.*
- Recommended (background for logic):
  Martin Goldstern, Haim Judah, *The Incompleteness Phenomenon. A New Course in Mathematical Logic.*

First-order languages and logic

First-order language(s)

We want to be as general as possible (logic after all is supposed to be universal). We define a language that we can use for (almost) anything. For the purpose, we select:

- a countable set $\mathrm{Var} = \{v_i \colon i \in \omega\}$ of variables
- a set $\{c_i \colon i \in \kappa_c\}$ of constant symbols
- for each nonzero arity n, a set $\{f_i^n \colon i \in \kappa_f\}$ of function symbols of arity n
- for each nonzero arity n, a set $\{P_i^n \colon i \in \kappa_P\}$ of relation symbols, (also called predicates), of arity n, among which there is a distinguished binary one, denoted by $=$.
- quantifiers: $\forall$, $\exists$
- Logical connectives: $\wedge, \vee, \rightarrow, \leftrightarrow, \neg$.
- Parentheses: (and).

κ_c, κ_f and κ_P are some cardinals, finite or infinite. Let us call this langauage $\mathcal{L}$.

Terms and formulae

A term of $\mathcal{L}$ is defined inductively as follows:

- every variable is a term
- every constant symbol is a term
- if f is an n-ary function symbol and $t_1, \ldots, t_n$ are terms, then $f(t_1, \ldots, t_n)$ is a term
- nothing else is a term

An atomic formula of $\mathcal{L}$ is (a string of the form) $P(t_1, \ldots, t_n)$, where P is an n-ary relation symbol, and $t_1, \ldots, t_n$ are terms. This includes formulae of the form $t = s$, for terms t and s. A formula of $\mathcal{L}$ is defined inductively as follows:

- every atomic formula is a formula
- if φ is a formula, and v a variable, then $\forall v)\varphi$, $\exists v\varphi$ and $\neg\varphi$ are formulae
- if φ, ψ are formulae, and $\star \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$, then $\varphi \star \psi$ is a formula
- nothing else is a formula

Free and bound variables, substitution, sentences

For a term t (or a formula φ) we write $\mathrm{Var}(t)$ (or $\mathrm{Var}(\varphi)$) for the set of variables occurring in t (or φ). An occurrence of v in φ is:

- bound, if it is within the scope of $\forall v$ or $\exists v$,
- free, otherwise.

We can substitute terms for free variables. Notation: $\varphi(x/t)$ is the result of substituting a term t for a free variable x. Substitution has to be handled with care; for example, let $\varphi(x)$ be $(\exists y)\ x \neq y$. Consider two substitutions:

- $\varphi(x/z)$, that is, $(\exists y)\ z \neq y$: no problem here.
- $\varphi(x/y)$, that is, $(\exists y)\ y \neq y$: BAD - a free x is captured.

To avoid capture of free variables, we disallow such bad substitutions. In general, a term t is not allowed for a variable v in a formula φ, if t contains a variable that becomes bound after substitution. Otherwise it is allowed.

- A formula with no free variables is called a sentence.

Consequence operators

We will denote the set of all formulae of $\mathcal{L}$ by For. We now define a logic in an abstract way. First, let Cn be a closure operator on For. That is, $\mathrm{Cn}\colon \wp(\mathrm{For}) \to \wp(\mathrm{For})$, such that

- $X \subseteq \mathrm{Cn}(X)$
- $X \subseteq Y \Rightarrow \mathrm{Cn}(X) \subseteq \mathrm{Cn}(Y)$
- $\mathrm{Cn}(\mathrm{Cn}(X)) = \mathrm{Cn}(X)$

This defines a closure operator. If Cn is moreover invariant under substitution, that is, if

- $\varphi \in \mathrm{Cn}(X)$ implies $\sigma(\varphi) \in \mathrm{Cn}(\sigma(X))$, for any substitution σ.

then Cn is structural. Further, if Cn also satisfies

- $\mathrm{Cn}(X) = \bigcup\{\mathrm{Cn}(Y)\colon Y \subseteq_{fin} X\}$

then it is finitary (or compact, or algebraic). A logic then, is the set $\mathrm{Cn}(\emptyset)$ for a structural consequence operator.

A Hilbert system for first-order logic

The commonest way of specifying Cn is via axioms and rules of inference. For example: the axioms are any formulae that are instantiations of the following schemes, as well as universal quantifications thereof:

A1. $\varphi \to (\psi \to \varphi)$.

A2. $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$.

A3. $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$.

A4. $\forall v \varphi \to \varphi(v/t)$, where t is allowed for v in φ.

A5. $\varphi \to \forall v \varphi$, where v is not free in φ.

A6. $\forall v(\varphi \to \psi) \to (\forall v \varphi \to \forall v \psi)$.

E1. $v = v$.

E2. $v = w \to w = v$.

E3. $v = w \wedge w = z \to v = z$.

E4. $v_1 = w_1 \wedge \cdots \wedge v_n = w_n \to f(v_1, \ldots, v_n) = f(w_1, \ldots, w_n)$ for any function symbol f.

E5. $v_1 = w_1 \wedge \cdots \wedge v_n = w_n \to R(v_1, \ldots, v_n) \leftrightarrow R(w_1, \ldots, w_n)$ for any relation symbol R.

Proofs

The only rule of inference is

R1. From φ and $\varphi \to \psi$, infer ψ. (Modus Ponens)

Other connectives are viewed as abbreviations: $\varphi \vee \psi$ is $\neg\varphi \to \psi$, $\varphi \wedge \psi$ is $\neg(\varphi \to \neg\psi)$, $\varphi \leftrightarrow \psi$ is $(\varphi \to \psi) \wedge (\psi \to \varphi)$, $\exists v \varphi$ is $\neg\forall v \neg\varphi$.

Let T be a set of formulae (a theory). Then, $\varphi \in \mathrm{Cn}(T)$ (commonly written as $T \vdash \varphi$), if there is a proof of φ from (assumptions in) T, that is a finite sequence of formulae $\psi_1, \ldots, \psi_n$ such that:

- ψ_n is φ
- for each $i < n$, ψ_i is
  - an axiom, or
  - a member of T, or
  - a result of applying an inference rule (Modus Ponens, in our case) to some formulae preceding ψ_i.

Compactness, and other theorems on provability

Theorem (Compactness)

Let T be a theory and φ a formula. Then, if $T \vdash \varphi$, then $T_0 \vdash \varphi$ for some finite $T_0 \subseteq T$.

Theorem (Deduction Theorem)

Let $T \cup \{\varphi, \psi\}$ be a set of formulae. The following equivalences hold:

1. $T \cup \{\varphi\} \vdash \psi$ *iff* $T \vdash \varphi \rightarrow \psi$
2. $T \cup \{\varphi\} \vdash \chi \wedge \neg\chi$ *for some formula* χ *iff* $T \vdash \neg\varphi$
3. $T \cup \{\neg\varphi\} \vdash \chi \wedge \neg\chi$ *for some formula* χ *iff* $T \vdash \varphi$

Theorem (Proof by cases)

Let $T \cup \{\varphi, \psi, \chi\}$ be a set of formulae. The following are equivalent:

1. $T \cup \{\varphi\} \vdash \chi$ *and* $T \cup \{\psi\} \vdash \chi$
2. $T \cup \{\varphi \vee \psi\} \vdash \chi$.

A theorem on constants

Theorem (Proof by arbitrary constant)

Let $\varphi(v)$ be a formula with exactly one free variable v, in some first-order language $\mathcal{L}$, and T be a set of formulae of $\mathcal{L}$. Let $\mathcal{L}^c$ be the expansion of $\mathcal{L}$ by a new constant symbol c. The following are equivalent:

1. $T \vdash \varphi(c)$
2. $T \vdash \forall v \varphi(v)$.

Proof.

Key observation for (1)→(2). Since c does not occur in any formula of T, do the following: whenever $\psi(c)$ occurs in the proof replace it by $\forall x \psi(x)$, with x a variable x that does not occur anywhere in the proof. Check that this is still a proof: axioms remain axioms (notice that if $\psi(c)$ is an axiom, so is $\psi(x)$ and therefore, so is $\forall x \psi(x)$), (ii) applications of modus ponens remain applications of modus ponens. The last formula of this proof is $\forall x \varphi(x)$. Rename the variables throughout to get $\forall v \varphi(v)$. □

Models

The semantic side of things is about models. A model $\mathcal{M}$, also called a structure or a relational structure is

$$\mathcal{M} = \langle M, (R_i)_{i \in I}, (f_j)_{j \in J}, (c_k)_{k \in K} \rangle$$

where

1. M is a nonempty set (the universe of the model).
2. $R_i \subseteq M^n$ is an n-ary relation, for each $i \in I$.
3. $f_j \colon M^m \to M$ is an m-ary function (an operation), for each $j \in J$.
4. c_k is an element of M (a constant), for each $k \in K$.

It is (almost) always assumed that among the relations there is a distinguished binary relation $=$ of 'true identity'. Constants, functions and relations together form the signature of $\mathcal{M}$, also called the language of of $\mathcal{M}$. (This is not quite precise: explain.)

Interpretation and valuation

Models are so called because they model formulae and theories of some language. Let $\mathcal{L}$ be a language, and $\mathcal{M}$ a structure. Suppose there is a function I (called interpretation), which assigns:

- to each constant c of $\mathcal{L}$ and element $c^{\mathcal{M}}$ of M,
- to each f of arity n a total function (written $f^{\mathcal{M}}$) from M^n into M,
- to each relation symbol R of arity n an n-ary relation over $\mathcal{M}$ (written $R^{\mathcal{M}}$).

If such a function I exists, $\mathcal{M}$ is an $\mathcal{L}$-structure: a structure appropriate for $\mathcal{L}$. In practice, the interpretation function is almost always implicit.

A valuation over domain M is a function ν assigning to each variable x an element of M. Valuations extend to compound terms by: $\nu(f(t_1 \ldots t_n)) = f^{\mathcal{M}}(\nu(t_1) \ldots \nu(t_n))$. (Valuations are homomorphisms!)

Satisfaction and truth definition

We say that a structure $\mathcal{M}$ satisfies atomic $R(t_1 \dots t_n)$ under a valuation ν, written $\mathcal{M} \models_\nu R(t_1 \dots t_n)$, or that a valuation ν satisfies $R(t_1 \dots t_n)$ in $\mathcal{M}$ if $\langle \nu(t_1) \dots \nu(t_n) \rangle \in P^{\mathcal{M}}$. Satisfaction extends to compound formulae, by the clauses below:

- $\mathcal{M} \models_\nu \varphi \to \psi$ if either $\mathcal{M} \not\models_\nu \varphi$ or $\mathcal{M} \models_\nu \psi$,
- $\mathcal{M} \models_\nu \neg\varphi$ if $\mathcal{M} \not\models_\nu \varphi$,
- $\mathcal{M} \models_\nu \forall v \varphi$ if for all valuations μ that differ from ν only at v, we have $\mathcal{M} \models_\mu \varphi$.

Such a μ is sometimes called a v-variant of ν.

- φ is true in $\mathcal{M}$ (written $\mathcal{M} \models \varphi$) if it is satisfied by all valuations.
- for a set T formulae, $\mathcal{M} \models T$ means $\mathcal{M} \models \varphi$ for every $\varphi \in T$.
- $T \models \varphi$ if $\mathcal{M} \models T$ implies $\mathcal{M} \models \varphi$, for every structure $\mathcal{M}$.

Language, structure, interpretation, valuation: an example

Language:

- constants: Z, U
- unary function: $m(\ )$
- binary functions: $p(\ ,\ )$, $t(\ ,\ )$
- binary relation: $S(\ ,\ )$

Structure: $\mathcal{R} = (\mathbb{R}; \leq, +, \cdot, -, 0, 1)$ the usual reals.

Interpretation: $Z^{\mathcal{R}}$ is 0, $U^{\mathcal{R}}$ is 1, $m^{\mathcal{R}}$ is $-$, $p^{\mathcal{R}}$ is $+$, $t^{\mathcal{R}}$ is $\cdot$, $S^{\mathcal{R}}$ is $\leq$, and $=^{\mathcal{R}}$ is $=$, as always. (Which is why I do not write $Z^{\mathcal{R}} = 0$, for example.)

A valuation: $v(x) = 7$, $v(y) = 13$, $v(z) = 90$, $\ldots$, so

$$\begin{aligned} v(p(Z, t(m(x), y))) &= v(Z) + v(t(m(x), y)) \\ &= 0 + (v(m(x)) \cdot v(y)) \\ &= 0 + (-v(x) \cdot 13) \\ &= 0 + (-7 \cdot 13) \\ &= -91 \end{aligned}$$

The example continued: satisfaction and truth

Let φ be the formula $\exists z\colon S(m(z), p(Z, t(m(x), y)))$.

Question

Does $\mathcal{R}$ satisfy φ under the valuation v? That is: $\mathcal{R} \models_v \varphi$?

By definition, $\mathcal{R} \models_v \exists z\colon S(m(z), p(Z, t(m(x), y)))$ if for some valuation v' differing from v at most on z (a z-variant of v) we have

$$\mathcal{R} \models_{v'} S(m(z), p(Z, t(m(x), y))).$$

Now, $S(m(z), p(Z, t(m(x), y)))$ is atomic, so that happens if

$$v'(m(z)) \leq v'(p(Z, t(m(x), y))).$$

We know that $v'(p(Z, t(m(x), y))) = -91$ (why?), and we now that $v'(m(z)) = -v'(z)$, so the question is: can we find a v' such that

$$-v'(z) \leq -91$$

Yes. Take $v'(z) = 100$, for example.

The example continued: truth

The formula φ, written in the usual way, would be

$$\exists z\colon\ -z \leq 0 + (-x \cdot y)$$

Question

Is φ true in $\mathcal{R}$? That is: $\mathcal{R} \models \varphi$?

$\mathcal{R} \models \varphi$ iff for every valuation v we have $\mathcal{R} \models_v \varphi$
iff for every valuation v there is
a z-variant v' of v such that $\mathcal{R} \models_{v'} \varphi$
... such that $-v'(z) \leq 0 + (-v'(x) \cdot v'(y))$
iff for every $a, b \in \mathbb{R}$ there is a $c \in \mathbb{R}$ with $-c \leq 0 + (-a \cdot b)$.

Question

Is the formula $\forall x, y\colon \varphi$ true in $\mathcal{R}$?

Another example

Consider the following theory T in the language of one binary relation:

1. $\forall x\ \exists y\colon R(x,y)$.
2. $\forall x,y\colon R(x,y) \to \neg R(y,x)$.
3. $\forall x,y\colon R(x,y) \to \exists z\colon R(x,z) \wedge R(z,y)$.

And the structure $\mathcal{M} = (\mathbb{Z}_7; R^{\mathcal{M}})$, where

$$R^{\mathcal{M}}(a,b) \text{ iff } a+1=b \text{ or } a+2=b \text{ or } a-3=b \pmod 7.$$

Question

Is T true in $\mathcal{M}$? YES! Draw a picture.

Completeness

Recall that we write $T \models \varphi$ if $\mathcal{M} \models T$ implies $\mathcal{M} \models \varphi$, for every structure $\mathcal{M}$.

Question

Is $T \vdash \varphi$ the same as $T \models \varphi$?

Theorem (Completeness theorem: Gödel, Mal'cev)

$T \vdash \varphi$ *iff* $T \models \varphi$

A set T of first-order formulae is said to be consistent if $T \vdash \varphi \wedge \neg\varphi$ holds for no formula φ (that is, if there is no formal proof of contradiction from assumptions in T). We say that T is satisfiable if there is a structure $\mathcal{M}$ and a valuation ν such that $\mathcal{M} \models_\nu \tau$, for every $\tau \in T$.

Theorem (Completeness theorem)

T is consistent iff T is satisfiable.

Henkin method

The difficult part in proving completeness is to show that if T is consistent, then it has a model. We call a theory S a Henkin theory, if

- S is maximally consistent (both consistent and complete)
- $\exists v \varphi \in S$ implies $\varphi(c) \in S$, for some constant c.

Note that $\varphi(c) \to \exists v \varphi$ is a logical truth, by axiom A4, so we have $\exists v \varphi \leftrightarrow \varphi(c) \in S$ (i.e., quantifiers are redundant).

Lemma

Let S be a Henkin theory. Then, S induces an equivalence relation on the set Tm of all terms (of the language of S), by putting $t \sim s$ if $t = s$ belongs to S.

Define the universe, functions and relations:

- $U = Tm/{\sim}$.
- $f^U([t_1], \ldots, [t_n]) = [f(t_1, \ldots, t_n)]$.
- $([t_1], \ldots, [t_n]) \in R^U$ iff $R(t_1, \ldots, t_n) \in S$.

Next, define a valuation ν:

- $\nu(v) = [v]$.

Henkin construction

Lemma

Let $\mathcal{U}$ and ν be the structure and the valuation defined above. Then $\mathcal{U} \models_\nu \varphi$, for every formula $\varphi \in S$.

Lemma (2 in 1: Lindenbaum+Henkin)

Let $\mathcal{L}$ be a language. Every consistent $\mathcal{L}$-theory T can be extended to a Henkin theory T^* in a language $\mathcal{L}^* = \mathcal{L} \cup C$, where C is a countable set of fresh constants.

A very rough sketch! Order all formulae into a sequence, and consider them one by one. Put $T_0^0 = T$.

- If $T_n^0 \cup \{\varphi_{n+1}\}$ is inconsistent, then $T_{n+1}^0 = T_n^0$.
- If $T_n^0 \cup \{\varphi_{n+1}\}$ is consistent, and φ_{n+1} is not of the form $(\exists v)\psi$, then $T_{n+1}^0 = T_n^0 \cup \{\varphi_{n+1}\}$.
- If $T_n^0 \cup \{\varphi_{n+1}\}$ is consistent, and φ_{n+1} is of the form $(\exists v)\psi$, then $T_{n+1}^0 = T_n^0 \cup \{\varphi_{n+1}, \psi(v/c)\}$, for some new constant c.
- $T_\omega^1 = \bigcup_{n\in\omega} T_n^0$.
- Repeat infinitely many times to get T^*.

Corollaries of Henkin construction

Theorem (Lower Löwenheim-Skolem Theorem)

Let T be a first order theory in a signature τ of cardinality κ. If T has an infinite model, then T has a model of cardinality $\max\{\aleph_0, \kappa\}$.

Sometimes called Löwenheim-Skolem paradox.

Theorem (Compactness)

A theory T has a model iff every finite subset of T has a model.

Proof.

For the nontrivial direction, suppose T has no model, but every finite $T_0 \subseteq T$ does. Then, by completeness, T is inconsistent, so $T \vdash \varphi \wedge \neg\varphi$ for some φ. But any proof of $\varphi \wedge \neg\varphi$ from T, is also a proof of $\varphi \wedge \neg\varphi$ from a finite $T_0 \subseteq T$. Thus, T_0 is inconsistent, and so, by completeness again, T_0 has no model, contradicting the assumption. □

Applications of compactness: infinity

Theorem (It takes infinity to recognise infinity)

Let ψ be a first-order sentence in pure equality language. If ψ holds in all infinite pure equality structures (sets), then there is an $n \in \mathbb{N}$ such that ψ holds in all sets S with $|S| > n$.

Proof.

Let $\Sigma_n = \{\neg\psi, \neg\varphi_1, \ldots, \neg\varphi_n\}$, where φ_n are the sentences from Assignment 1 Question 4. Suppose each Σ_n has a model. This model is a set with at least $n+1$ elements. Put $\Sigma = \bigcup_{n\in\mathbb{N}} \Sigma_n$. By compactness, Σ has a model, say, S. Then, for each $n \in \mathbb{N}$ we have that S has strictly more elements than n, so S is infinite. Yet, $S \models \neg\psi$. Contradiction. □

Corollary

The class $\mathbb{INF}$ of infinite pure equality structures is not finitely axiomatisable.

But it is an elementary class. It is axiomatised by some set of first-order sentences, namely, $\{\neg\varphi_n : n \in \mathbb{N}\}$.

Applications of compactness: Robinson's principle

Theorem (Robinson's principle)

Let φ be a first-order sentence in the language of fields. If φ holds in all fields of characteristic 0, then there is a prime p such that φ holds in all fields of characteristic $\geq p$.

Proof.

By contradiction. Let Φ be some first-order rendering of field axioms, and for each prime p let χ_p be the sentence $\underbrace{1+1+\cdots+1}_{p \text{ times}} \neq 0$. Let $\Delta_p = \{\chi_q \colon q \leq p\}$.

Now, suppose there is an infinite sequence of primes $(p_i)_{i\in I}$, such that each set $\Sigma_i = \{\neg\varphi\} \cup \Delta_{p_i} \cup \Phi$, has a model. By compactness, $\Sigma = \bigcup_{i\in I} \Sigma_i$ has a model, say, M. Then, M is a field of characteristic 0, and $M \models \neg\varphi$. But $M \models \varphi$ by assumption. Contradiction. □

Applications of compactness: torsion-free groups

Definition

A group $\mathbf{G}$ is a torsion group, if for every $g \in G$ we have $g^n = e$ for some n. A group $\mathbf{G}$ is a torsion free group, if for every $g \in G \setminus \{e\}$ we have $g^n \neq e$ for every n.

Theorem

The class $\mathbb{TF}$ of all torsion-free groups is not finitely axiomatisable. That is, there is no finite set S of first-order formulae, such that $\mathbf{G} \in \mathbb{TF}$ iff $\mathbf{G} \models S$.

Proof.

Observe that $\mathbf{G}$ is torsion-free iff $\mathbf{G} \models \Sigma$, where $\Sigma = \{x \neq e \rightarrow x^n \neq e \colon n \in \mathbb{N}\}$. Suppose, towards a contradiction, that S is a finite set axiomatising $\mathbb{TF}$. Let $\sigma = \bigwedge S$. Then, $\Sigma \models \sigma$, that is, $\Sigma \cup \{\neg\sigma\}$ has no model. Therefore, $\Sigma_0 \cup \{\neg\sigma\}$ has no model, for some finite $\Sigma_0 \subseteq \Sigma$. Take a group $\mathbf{H}$ such that: (i) $\mathbf{H}$ is not torsion-free, and (ii) all elements of $\mathbf{H}$ are of order big enough to satisfy Σ_0. Contradiction. □

Homomorphisms,
substructures,
elementary substructures,
elementary equivalence,
direct products

Homomorphisms

Let the structures below be structures in the same language (terminology: of the same signature, or similar)

$$\mathcal{A} = \langle A, (R_i)_{i \in I}, (f_j)_{j \in J}, (c_k)_{k \in K} \rangle \text{ and}$$
$$\mathcal{B} = \langle B, (S_i)_{i \in I}, (g_j)_{j \in J}, (d_k)_{k \in K} \rangle$$

A map $h\colon A \to B$ such that:

- $h(c_k) = d_k$, for every $k \in K$
- $h(f_j(\overline{a})) = g_j(h(\overline{a}))$, for every $j \in J$ and every tuple (of appropriate length) $\overline{a}$ from A.
- $\overline{a} \in R_i$ implies $h(\overline{a}) \in S_i$, for every $j \in J$ and every tuple (of appropriate length) $\overline{a}$ from A.

is a homomorphism from $\mathcal{A}$ to $\mathcal{B}$. If moreover h is injective and

- $\overline{a} \in R_i$ if and only if $h(\overline{a}) \in S_i$, for every $j \in J$ and every tuple (of appropriate length) $\overline{a}$ from A.

then h is an embedding. A surjective embedding is an isomorphism. We write $\mathcal{A} \cong \mathcal{B}$ for '$\mathcal{A}$ is isomorphic to $\mathcal{B}$'.

Substructures

Let

$$\mathcal{A} = \langle A, (R_i)_{i \in I}, (f_j)_{j \in J}, (c_k)_{k \in K} \rangle \text{ and}$$
$$\mathcal{B} = \langle B, (S_i)_{i \in I}, (g_j)_{j \in J}, (d_k)_{k \in K} \rangle$$

be similar structures, such that

- $A \subseteq B$
- the inclusion map $i \colon A \to B$ is an embedding

then $\mathcal{A}$ is a substructure of $\mathcal{B}$ (written $\mathcal{A} \leq \mathcal{B}$). Notice that this implies:

- $c_k = d_k$ for every $k \in K$ (A contains all constants from B)
- $f_j = g_j|_A$ (A is closed under the operations)
- $R_i = S_j|_A$ (relations on A are restrictions of relations on B)

Conversely, if all these hold, then $\mathcal{A} \leq \mathcal{B}$.
For $X \subseteq B$, we define the substructure generated by X (sometimes denoted by $\langle X \rangle_B$) in the natural inductive way.

Elementary substructures and elementary equivalence

- Similar structures $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent (written $\mathcal{A} \equiv \mathcal{B}$) if for every sentence φ, in the appropriate language, $\mathcal{A} \models \varphi$ iff $\mathcal{B} \models \varphi$.
- $\mathcal{A}$ is an elementary substructure of $\mathcal{B}$ (written $\mathcal{A} \prec \mathcal{B}$) if $\mathcal{A} \leq \mathcal{B}$ and, for every formula $\varphi(x_1, \ldots, x_n)$ with the free variables displayed, and every valuation ν such that $\nu(x_i) = a_i \in A$ for every $1 \leq i \leq n$, we have $\mathcal{A} \models_\nu \varphi$ iff $\mathcal{B} \models_\nu \varphi$.
- Equivalently, $\mathcal{A}$ is an elementary substructure of $\mathcal{B}$ if, for every formula $\varphi(a_1, \ldots, a_n)$ with parameters $a_1, \ldots, a_n$ from A, we have $\mathcal{A} \models \varphi$ iff $\mathcal{B} \models \varphi$.

Lemma (Tarski-Vaught test)

Let $\mathcal{A}$ and $\mathcal{B}$ be similar structures such that $\mathcal{A} \leq \mathcal{B}$. Then, $\mathcal{A} \prec \mathcal{B}$ holds, if for every formula $\varphi(x, a_1, \ldots, a_n)$ with parameters $a_1, \ldots, a_n$ from A, we have that $\mathcal{B} \models \exists x \varphi$ implies $\mathcal{A} \models \exists x \varphi$.

Elementary equivalence and isomorphism

Lemma

Let $\mathcal{A}$ and $\mathcal{B}$ be similar structures. Then, $\mathcal{A} \cong \mathcal{B}$ implies $\mathcal{A} \prec \mathcal{B}$ and $\mathcal{A} \prec \mathcal{B}$ implies $\mathcal{A} \equiv \mathcal{B}$. The converses are not true.

Example

Let $\mathcal{A}$ and $\mathcal{B}$ be infinite pure identity structures. Suppose A is a proper subset of B, of stricly smaller cardinality (for example, $B = \wp(A)$. Then, $\mathcal{A} \prec \mathcal{B}$, but $\mathcal{A} \ncong \mathcal{B}$. Moreover, since $\mathcal{A} \prec \mathcal{B}$ implies $\mathcal{A} \equiv \mathcal{B}$, and $\equiv$ is symmetric, we have $\mathcal{B} \equiv \mathcal{A}$, but $\mathcal{B} \nprec \mathcal{A}$. Notice that finite ones will not do.

Warning: Two different meanings of $\prec$

Let $\mathcal{A}$ and $\mathcal{B}$ be infinite pure identity structures. If $\mathcal{A} \prec \mathcal{B}$ is taken as implying $\mathcal{A} \leq \mathcal{B}$, then $\mathcal{A} \prec \mathcal{B}$ and $\mathcal{B} \prec \mathcal{A}$ trivially imply $\mathcal{A} \cong \mathcal{B}$. If $\mathcal{A} \prec \mathcal{B}$ is taken as '$\mathcal{A}$ is elementarily embeddable into $\mathcal{B}$', then $\mathcal{A} \prec \mathcal{B}$ and $\mathcal{B} \prec \mathcal{A}$ still imply $\mathcal{A} \cong \mathcal{B}$, but only via Schröder-Bernstein Theorem.

Direct products

Let $(\mathcal{A}_i)_{i\in I}$ be a family of similar structures. The direct product $\mathcal{A} = \prod_{i\in I} \mathcal{A}_i$ is a structure such that

- constants, functions and relations of $\mathcal{A}$ are defined coordinatewise
- in particular, a tuple $\overline{a} = \langle a_0, \dots, a_n \rangle$ belongs to a relation R^A, if $\langle a_0(i), \dots, a_n(i) \rangle \in R^{A_i}$ at every coordinate $i \in I$.

Note that this is equivalent to:

- $\langle a_0, \dots, a_n \rangle \in R^A$ iff $\{i \in I\colon \langle a_0(i), \dots, a_n(i) \rangle \in R^{A_i}\} = I$

A clever weakening if the condition on the right-hand side will lead to the definition of a reduced product, and ultraproduct.

Diagrams

Parameters made precise

Let $\mathcal{A}$ be a structure, $\overline{a}$ be a sequence of elements of A, and $\varphi(\overline{x})$ a formula with free variables $\overline{x}$. We use $\mathcal{A} \models \varphi[\overline{a}]$ to mean $\mathcal{A} \models_\nu \varphi(\overline{x})$ such that $\nu(\overline{x}) = \overline{a}$, and said: This is just using elements of A as parameters.

This device is so often used that it got systematised.

- We write $(\mathcal{A}, \overline{a})$ for the structure with the same universe, relations and functions, but with the signature expanded by a sequence of constants $\overline{c}$, such that $c_i^{(\mathcal{A},\overline{a})} = a_i$.
- If the original signature is L, we write $L(\overline{c})$ for the new signature.
- Very often, we interested in the substructure of $(\mathcal{A}, \overline{a})$ generated by $\overline{a}$, which we denote by $\langle \overline{a} \rangle_{\mathcal{A}}$ with the subscript dropped if $\mathcal{A}$ is clear from context.

Diagrams

Lemma (trivial, and used silently)

Let $\mathcal{A}$, $\mathcal{B}$ be structures in signature L, and $(\mathcal{A}, \bar{a})$, $(\mathcal{B}, \bar{b})$ be structures of signature $L(\bar{c})$. Then, a homomorphism (embedding) $f\colon (\mathcal{A}, \bar{a}) \to (\mathcal{B}, \bar{b})$ is the same thing as a homomorphism (embedding) $f\colon \mathcal{A} \to \mathcal{B}$ with $f(\bar{a}) = \bar{b}$.

Recall that an atomic sentence is an atomic formula which is a sentence, that is, has no free variables. Atomic sentences exist, if the signature has constants.

Definition

- A diagram of $\mathcal{A}$, written $\mathrm{diag}(\mathcal{A})$, is the set of all atomic sentences and their negations true in $(\mathcal{A}, \bar{a})$.
- A positive diagram of $\mathcal{A}$, written $\mathrm{diag}^+(\mathcal{A})$, is the set of all atomic sentences true in $(\mathcal{A}, \bar{a})$.

Properties of diagrams

Lemma (Diagram lemma)

Let $\mathcal{A}$ and $\mathcal{B}$ be L-structures, $\overline{c}$ a sequence of constants, and $(\mathcal{A}, \overline{a})$, $(\mathcal{B}, \overline{b})$ be $L(\overline{c})$-structures. The following are equivalent.

1. *For every atomic sentence φ, if $(\mathcal{A}, \overline{a}) \models \varphi$, then $(\mathcal{B}, \overline{b}) \models \varphi$.*
2. *There is a homomorphism $f : \langle \overline{a} \rangle_{\mathcal{A}} \to (\mathcal{B}, \overline{b})$, such that $f(\overline{a}) = \overline{b}$.*
3. $\mathcal{B} \models \mathrm{diag}^{+}(\langle \overline{a} \rangle_{\mathcal{A}})$

If it exists, the homomorphism f is unique. Moreover, t.f.a.e.

- *f is an embedding*
- *for every atomic sentence φ, $(\mathcal{A}, \overline{a}) \models \varphi \Leftrightarrow (\mathcal{B}, \overline{b}) \models \varphi$*
- $\mathcal{B} \models \mathrm{diag}(\langle \overline{a} \rangle_{\mathcal{A}})$

Proof of diagram lemma: $1 \Rightarrow 2$

- Assume 1. Take an atomic sentence φ and suppose $(\mathcal{A}, \overline{a}) \models \varphi$. Because φ is atomic and has no free variables, it must be of the form $R(t_1, \ldots, t_n)$ for constant terms $t_1, \ldots, t_n$. So, each $t_i^{(\mathcal{A},\overline{a})}$ is of the form $t_i(\overline{a}, \overline{d})$, where $\overline{d}$ is a (possibly empty) tuple of constants from the signature L.
- Therefore, $R^{(\mathcal{A},\overline{a})}(t_1, \ldots, t_n)$ holds, and so $R^{\mathcal{A}}(t_1, \ldots, t_n)$ holds, too.
- Furthermore, the elements $t_i^{(\mathcal{A},\overline{a})}$ belong to $\langle \overline{a} \rangle_{\mathcal{A}}$, and each element of $\langle \overline{a} \rangle_{\mathcal{A}}$ is of that form.
- We define a map $f \colon \langle \overline{a} \rangle_{\mathcal{A}} \to B$ putting $f(t_i^{(\mathcal{A},\overline{a})}) = t_i^{(\mathcal{B},\overline{b})}$, choosing some representative term for each element of $\langle \overline{a} \rangle_{\mathcal{A}}$. This does not depend on the choice of representatives, because $t^{(\mathcal{A},\overline{a})} = s^{(\mathcal{A},\overline{a})}$ implies $(\mathcal{A}, \overline{a}) \models t = s$, and so, by 1, $(\mathcal{B}, \overline{b}) \models t = s$; hence $t^{(\mathcal{B},\overline{b})} = s^{(\mathcal{B},\overline{b})}$.
- Then $f(\overline{a}) = \overline{b}$. That f is a homomorphism follows from definition of satisfaction.

Proof of diagram lemma: 2 $\Rightarrow$ 3 $\Rightarrow$ 1

- Assume 2. By definition, $\mathrm{diag}^+(\langle \overline{a} \rangle_\mathcal{A})$ is the set of all atomic sentences true in $\langle \overline{a} \rangle_\mathcal{A}$.
- Since $\langle \overline{a} \rangle_\mathcal{A} \leq (\mathcal{A}, \overline{a})$, we get that $(\mathcal{A}, \overline{a}) \models \mathrm{diag}^+(\langle \overline{a} \rangle_\mathcal{A})$.
- By 2, f is a homomorphism, and since all sentences in $\mathrm{diag}^+(\langle \overline{a} \rangle_\mathcal{A})$ are positive, we get $\mathcal{B} \models \mathrm{diag}^+(\langle \overline{a} \rangle_\mathcal{A})$, which is what 3 states.
- Now, assume 3, and take any atomic sentence φ, such that $(\mathcal{A}, \overline{a}) \models \varphi$.
- Then, φ is a member of $\mathrm{diag}^+(\langle \overline{a} \rangle_\mathcal{A})$, and so, by 2, $(\mathcal{B}, f(\overline{a}) \models \varphi$. Since $f(\overline{a}) = \overline{b}$ the claim is proved.

Corollary

For any structures $\mathcal{A}$ and $\mathcal{B}$ of the same signature, t.f.a.e.

- *$\mathcal{A}$ is isomorphic to $\mathcal{B}$*
- $\mathcal{B} \models \mathrm{diag}(\mathcal{A})$ *and* $\mathcal{A} \models \mathrm{diag}(\mathcal{B})$

Preservation theorems

Normal forms

A quantifier-free formula φ is is in disjunctive normal form if it is a disjunction of conjunctions of atomic or negated atomic formulae.

Example

$x = y \to (P(x, z) \to x = y)$ is not in disjunctive normal form.
$\neg(x = y) \vee \neg P(x, z) \vee x = y$ is.

A formula φ is in prenex form if it is of the form $Q_1 \ldots Q_n \psi$, where $Q_1 \ldots Q_n$ is a string of quantifiers and ψ is quantifier-free.

Example

$\forall x \colon P(x) \to \exists x \colon Q(x, y)$ is not in prenex form.
$\forall x \; \exists z \colon P(x) \to Q(z, y)$ is is prenex form.
$\forall x \; \exists z \colon \neg P(x) \vee Q(z, y)$ is in prenex disjunctive normal form.

Lemma (prenex disjunctive normal form)

Every first-order formula is equivalent to one in prenex disjunctive normal form.

Stratification of formulae

Assume φ is in prenex disjunctive normal form.

- φ is positive if it does not contain occurrences of negation (more precisely, if every atomic subformula is within the scope of an even number of negations).
- φ is universal if it is of the form $(\forall v_1 \ldots \forall v_n)\psi$, and ψ is quantifier-free.
- φ is existential if it is of the form $(\exists v_1 \ldots \exists v_n)\psi$, and ψ is quantifier-free.

"Arithmetical" hierarchy:

- $\varphi \in \forall_0 = \exists_0$, if φ is quantifier-free.
- $\varphi \in \forall_{n+1}$, if φ is of the form $(\forall v_1 \ldots \forall v_n)\psi$, and $\psi \in \exists_n$.
- $\varphi \in \exists_{n+1}$, if φ is of the form $(\exists v_1 \ldots \exists v_n)\psi$, and $\psi \in \forall_n$.

Whenever we say that a formula φ is $\forall_k$ ($\exists_k$) we mean that φ is equivalent to an $\forall_k$ ($\exists_k$) formula.

Prenex disjunctive (conjunctive) normal form as a matrix

A formula $\varphi(\overline{x})$ in prenex disjunctive normal form can be visualised as something resembling a (ragged) matrix

$$\text{prefix of quantifiers} \begin{pmatrix} \pm\alpha^1_1(\overline{x}) & \pm\alpha^1_2(\overline{x}) & \dots \\ \pm\alpha^2_1(\overline{x}) & \pm\alpha^2_2(\overline{x}) & \dots \\ \dots & \vdots & \\ \pm\alpha^n_1(\overline{x}) & \pm\alpha^n_2(\overline{x}) & \dots \end{pmatrix}$$

with rows not neccessarily of the same length. Each $\alpha^i_j(\overline{x})$ is atomic, in variables from the list $\overline{x}$ (but not neccessarily all of them occurring in α^i_j). The $\pm$ sign in front means that each α^i_j is either negated or not negated. Rows are conjunctions, and the whole "matrix" is the disjunction of its rows.

The conjunctive normal form is the dual: rows are disjunctions and the whole thing is the conjunction of rows. The two forms are equivalent (by Boolean algebra).

Simple preservation theorems: substructures

Theorem (substructures)

Let $\mathcal{A}$ be a substructure of $\mathcal{B}$.

- *If $\varphi(\overline{x})$ is an existential formula and $\overline{a}$ is a tuple of elements of A, then $\mathcal{A} \models \varphi[\overline{a}]$ implies $\mathcal{B} \models \varphi[\overline{a}]$,*
- *If $\varphi(\overline{x})$ is a universal formula and $\overline{a}$ is a tuple of elements of A, then $\mathcal{B} \models \varphi[\overline{a}]$ implies $\mathcal{A} \models \varphi[\overline{a}]$.*

Proof.

The formula $\varphi(\overline{x})$ with parameters filled in, really looks like this $\forall \overline{y}\colon \psi(\overline{y}, \overline{a})$, where $\psi(\overline{y}, \overline{a})$ is quantifier-free in normal form. By definition of satisfaction, $\mathcal{B} \models \forall \overline{y}\colon \psi(\overline{y}, \overline{a})$, if for all tuples $\overline{b}$ taken from B, we have $\mathcal{B} \models \psi(\overline{b}, \overline{a})$. Now, take any tuple $\overline{c}$ taken from A. It is also a tuple from B, because $A \subseteq B$. So, clearly, $\mathcal{B} \models \psi(\overline{c}, \overline{a})$. What remains is to show that $\mathcal{A} \models \psi(\overline{c}, \overline{a})$. But, as $\mathcal{A} \leq \mathcal{B}$, for any relation R and function f in the signature we have $R^{\mathcal{A}} = R^{\mathcal{B}}|_A$ and $f^{\mathcal{A}} = f^{\mathcal{B}}|_A$. Now, look at the "matrix" of ψ, and see that it works. □

Simple preservation theorems: homomorphisms

Theorem (homomorphisms)

Let $\mathcal{A}$ and $\mathcal{B}$ be structures and $f : A \to B$ a homomorphism. If φ is a positive existential formula and $\bar{a}$ is a tuple of elements of A, then $\mathcal{A} \models \varphi[\bar{a}]$ implies $\mathcal{B} \models \varphi[f(\bar{a})]$.

Proof.

As before, φ really is $\exists \bar{y} \colon \psi(\bar{y}, \bar{a})$, so $\mathcal{A} \models \varphi[\bar{a}]$ iff for some $\bar{b}$ from A, we have $\mathcal{A} \models \psi(\bar{b}, \bar{a})$. Now, ψ is quantifier-free in normal form, and it is positive, so there are no negated atomic formulae in its "matrix". Take an atomic $\alpha^i_j(\bar{b}, \bar{a})$. Then, $\mathcal{A} \models \alpha^i_j(\bar{b}, \bar{a})$ implies $\mathcal{A} \models \alpha^i_j(f(\bar{b}), f(\bar{a}))$. The rest follows. □

Example

- "Positiveness" cannot be relaxed. Map $\mathbf{2} \times \mathbf{2}$ to $\mathbf{4}$ (as ordered sets). The first one satisfies $\exists x, y \colon (x \not\leq y) \wedge (y \not\leq x)$, the second does not.
- "Existentiality" can be dropped, if f is onto.

Preservation: Horn clauses and direct products

Definition

A universally quantified disjunction $\varphi = (\forall \dots)\ \psi_1 \vee \cdots \vee \psi_n$, such that each ψ_i is either atomic or negated atomic, and at most one of them is positive, is called a Horn clause. A Horn formula is a conjunction of Horns clauses.

Theorem (direct products)

Let $(\mathcal{A}_i)_{i \in I}$ be a family of similar structures, and $\mathcal{A} = \prod_{i \in I} \mathcal{A}_i$. Let φ be a Horn clause. If $\mathcal{A}_i \models \varphi$ for every $i \in I$, then $\mathcal{A} \models \varphi$. The same holds for Horn formulae.

Proof.

Wlog, φ is $\alpha_1 \vee \neg\alpha_2 \vee \cdots \vee \neg\alpha_n$. We have two cases. (1) α_1 holds at every coordinate $i \in I$. Then, α_1 holds in the product. (2) α_1 fails at some coordinate i_0. Then, some $\neg\alpha_k$ holds at coordinate i_0, and so α_k fails at i_0. Thus, α_k fails in the product, that is, $\neg\alpha_k$ holds in the product. In either case φ holds in the product. □

Preservation: unions of chains

Theorem (chains)

Let $(\mathcal{A}_i)_{i \in I}$ *be a family of structures that form a chain under embedding, that is,* $\mathcal{A}_0 \leq \mathcal{A}_1 \leq \ldots$ *. Put* $\mathcal{A} = \bigcup_{i \in I} \mathcal{A}_i$*. Let* $\varphi(\overline{x})$ *be a* $\forall_2$ *formula, and* $\overline{a}$ *a tuple from* A_0*. If* $\mathcal{A}_i \models \varphi[\overline{a}]$ *for every* $i \in I$*, then* $\mathcal{A} \models \varphi[\overline{a}]$*.*

Proof.

Write φ more explicitly, as $\forall \overline{y} \exists \overline{z} \colon \psi(\overline{x}, \overline{y}, \overline{z})$. Take any tuple $\overline{b}$ from A. Since $\overline{b}$ is finite it belongs to some A_i. Since $\mathcal{A}_i \models \varphi$ by assumption, we have that $\mathcal{A}_i \models \exists \overline{z} \colon \psi(\overline{a}, \overline{b}, \overline{z})$. Now that is an existential formula with parameters from A_i, and and $\mathcal{A}_i \leq \mathcal{A}$. So, by the preservation theorem on substructures, we conclude $\mathcal{A} \models \exists \overline{z} \colon \psi(\overline{a}, \overline{b}, \overline{z})$, proving the claim. □

Applications of diagrams: a straightforward one

Theorem (Łoś-Tarski)

Let T be a theory. If T is preserved under substructures, then T is equivalent to a set of $\forall_1$ formulae.

Proof sketch.

Wlog, T is consistent. Let $T_\forall$ be $\{\varphi\colon \varphi$ is $\forall_1$ and $T \vdash \varphi\}$. Let $\mathbb{K} = \{\mathcal{A}\colon \mathcal{A} \leq \mathcal{B}$ for some $\mathcal{B} \in \mathrm{Mod}(T)\}$. We will show that $\mathbb{K} = \mathrm{Mod}(T_\forall)$. This suffices, because we have $\mathrm{Mod}(T) \subseteq \mathbb{K} \subseteq \mathrm{Mod}(T_\forall)$, and thus, by assumption, also $\mathrm{Mod}(T) = \mathbb{K} \subseteq \mathrm{Mod}(T_\forall)$.

- For the converse, take $\mathcal{A} \models T_\forall$.
- Key observation: $\mathrm{diag}(\mathcal{A}) \cup T$ is consistent (proof by arbitrary constant!). Therefore it has a model, say, $\mathcal{M}$.
- Since $\mathcal{M} \models \mathrm{diag}(\mathcal{A})$, by the diagram lemma $\mathcal{A} \leq \mathcal{M}$.
- So, $\mathcal{A} \in \mathbb{K}$ as claimed.

□

Applications of diagrams: a fancier one

Theorem (Chang-Łoś-Suszko)

Let T be a theory. If T is preserved under unions of chains, then T is equivalent to a set of $\forall_2$ formulae.

How to start off.

Let $T_{\forall\exists}$ be $\{\varphi\colon \varphi$ is $\forall_2$ and $T \vdash \varphi\}$. We will show that every model of $T_{\forall\exists}$ is a model of T. Let $\mathcal{A} \models T_{\forall\exists}$. Expand the language, naming all elements of A. Let $D_\forall(\mathcal{A})$ be the set of all universal sentences in the expanded signature (in the diagram language of $\mathcal{A}$) which are true in $(\mathcal{A}, \bar{a})$.

- $D_\forall(\mathcal{A}) \cup T$ is consistent.
- Let $\mathcal{B} \models D_\forall(\mathcal{A}) \cup T$.
- Since $D_\forall(\mathcal{A}) \supseteq \mathrm{diag}(\mathcal{A})$, we have $\mathcal{A} \leq \mathcal{B}$.
- $\mathrm{diag}(\mathcal{B}) \cup \mathrm{Th}(\mathcal{A}, \bar{a})$ is consistent (this is taken in the diagram language of $\mathcal{B}$).
- Let $\mathcal{A}_1 \models \mathrm{diag}(\mathcal{B}) \cup \mathrm{Th}(\mathcal{A}, \bar{a})$.

□

Proof continued: alternating chains

How to get there.

- Then $\mathcal{A} \leq \mathcal{B} \leq \mathcal{A}_1$, and $\mathcal{B} \models T$.
- Moreover, $\mathcal{A} \prec \mathcal{A}_1$. The elementarity of the embedding follows from the fact that $\mathcal{A}_1 \models \mathrm{Th}(\mathcal{A}, \bar{a})$.
- Continuing inductively, we get $\mathcal{A} \leq \mathcal{B} \leq \mathcal{A}_1 \leq \mathcal{B}_1 \leq \mathcal{A}_2 \leq \dots$
- Let $\mathcal{C}$ be the union of this chain.
- Then, $\mathcal{C}$ is also the union of $\mathcal{B} \leq \mathcal{B}_1 \leq \mathcal{B}_2 \leq \dots$ and as T is preserved under unions of chains, we get $\mathcal{C} \models T$.
- But, $\mathcal{C}$ is also the union of $\mathcal{A} \leq \mathcal{A}_1 \leq \mathcal{A}_2 \leq \dots$, and as $\mathcal{A} \prec \mathcal{A}_1 \prec \mathcal{A}_2 \prec \dots$, we get $\mathcal{A} \prec \mathcal{C}$.
- Therefore, $\mathcal{A} \models T$, as required.

$\square$

Equality-closed sets of atomic sentences

Let $\mathcal{A}$ be a structure and T the set of all atomic sentences true in $\mathcal{A}$. The T has the following two properties:

1. For every closed term t, the atomic sentence $t = t$ is in T.
2. For any atomic formula $\varphi(x_1, \ldots, x_n)$, if the equations $s_1 = t_1, \ldots, s_n = t_n$ are in T, then $\varphi(\bar{s}) \in T$ iff $\varphi(\bar{t}) \in T$.

Definition

A set T of atomic sentences is equality-closed if it has the two properties above.

Lemma (SMT: Lemma 1.5.1)

Let T be an equality-closed set of atomic sentences of some signature L. Then, there exists an L-structure $\mathcal{A}$ such that

- *T is the set of all atomic sentences true in $\mathcal{A}$.*
- *Every element of A is of the form $t^{\mathcal{A}}$ for some closed term.*

Proof idea.

Repeat Henkin construction (with T instead of a Henkin set). □

Canonical models, aka zero-generated free algebras

Theorem (Canonical model)

Let T be a set of atomic sentences in some signature L. Then, there exists an L-structure $\mathcal{M}$ such that

1. $\mathcal{M} \models T$,
2. *every element of M is of the form t^M for a closed term t,*
3. *if $\mathcal{N}$ is an L-structure and $\mathcal{N} \models T$, then there exists a unique homomorphism $h \colon \mathcal{M} \to \mathcal{N}$.*

Proof.

Observe that for any set T of atomic sentences there is the smallest equality-closed set U of atomic sentences containing T. Apply the lemma from the previous slide to U. This proves (1) and (2). By (2) $\mathcal{M} = \langle\emptyset\rangle_{\mathcal{M}}$, so, to prove (3), we want to apply the diagram lemma. To do so, we need to show that $\mathcal{N} \models \mathrm{diag}^+(\mathcal{M})$. By construction $\mathcal{M} \models U$; moreover $\mathrm{diag}^+(\mathcal{M}) \subseteq U$. Let S be the set of all atomic sentences true in $\mathcal{N}$. Then, (a) S is equality-closed and (b) $T \subseteq S$. Therefore, $U \subseteq S$. $\square$

Adding roots of polynomials to a field

Let F be a field. We write $F[x]$ for the ring of polynomials over F in the indeterminant x. We can think of $F[x]$ as a structure in the signature $L = (+, -, \cdot, \{c_f \colon f \in F\}, c_x)$, that is, of rings with additional constants for each element of F, and one more for x. Take a polynomial $f(x)$ which is irreducible over F. Let T be the set of all equations that are true in $F[x]$. Now consider $T \cup \{f(x) = 0\}$. This is a set of atomic sentences, so it has a canonical model $\mathcal{M}$.

- By diagram lemma, there is an onto homomorphism $h \colon F[x] \to \mathcal{M}$. In particular $\mathcal{M}$ is a ring.
- As $\mathcal{M} \models f(x) = 0$, we have $h(a) = 0^{\mathcal{M}}$ for every a in the ideal generated by $f(x)$.
- Because $F[x]_{f(x)}$ satisfies $T \cup \{f(x) = 0\}$, we get that $F[x]_{f(x)}$ is a homomorphic image of $\mathcal{M}$.
- In fact, $F[x]_{f(x)}$ and $\mathcal{M}$ are isomorphic (consider $F[x] \longrightarrow \mathcal{M} \longrightarrow F[x]_{f(x)}$; composition is quotient map).

Ultraproducts

Filters and ultrafilters

Definition

A nonempty subset F of the universe of a Boolean algebra $\mathbf{B}$ is an filter if

- $a \in F$ and $a \leq b$ implies $b \in F$.
- $a, b \in F$ implies $a \wedge b \in F$.

A filter F is principal if it is of the form $\uparrow a$ for some $a \in B$.
A maximal, proper filter is called an ultrafilter.

Lemma

Let F be a filter on a Boolean algebra $\mathbf{B}$. The following are equivalent.

1. *F is an ultrafilter.*
2. *If $a \vee b \in F$ then $a \in F$ or $b \in F$.*
3. *$\neg a \in F$ iff $a \notin F$.*

A filter in $\mathcal{P}(X)$ for some set X is called a filter over X.

Finite intersection property, principal ultrafilters

Definition

Let X be a set, and $\mathcal{C}$ a collection of subsets of X. Then, $\mathcal{C}$ has finite intersection property if for every finite $C_1, \ldots, C_n \in \mathcal{C}$ we have $C_1 \cap \cdots \cap C_n \neq \emptyset$.

Lemma

Let X be a set, and $\mathcal{C}$ a collection of subsets of X. If $\mathcal{C}$ has finite intersection property, then it can be extended to an ultrafilter over X, that is, there exists an ultrafilter U over X such that $C \in U$ for every $C \in \mathcal{C}$.

Example

Consider $\mathbb{N}$ and let $\mathcal{C}$ be the set of all cofinite subsets of $\mathbb{N}$. Then $\mathcal{C}$ has the finite intersection property.

Lemma

An ultrafilter U over X is principal iff U is of the form $\uparrow\{x_0\}$ for some $x_0 \in X$.

Reduced products

Let $(\mathcal{A}_i)_{i\in I}$ be a family of similar structures. Let F be a filter on $\mathcal{P}(I)$ (also called a filter over I). On $\prod_{i\in I} A_i$ define a binary relation putting

$$a \sim b \quad \text{iff} \quad \{i \in I \colon a(i) = b(i)\} \in F$$

Lemma

The relation $\sim$ is an equivalence, preserving all functions of $\prod_{i\in I} \mathcal{A}_i$.

Define $B = \prod_{i\in I} A_i/_\sim$, put $f^B = f/_\sim$, $c^B = c/_\sim$, for every function f and constant c of $\prod_{i\in I} \mathcal{A}_i$. Finally, for every relation R of $\prod_{i\in I} \mathcal{A}_i$ and every tuple $\langle a_0, \ldots, a_n\rangle$ of elements of $\prod_{i\in I} A_i$, put

$$\langle a_0/_\sim, \ldots, a_n/_\sim\rangle \in R^B \quad \text{iff} \quad \{i \in I \colon \langle a_0(i), \ldots, a_n(i)\rangle \in R\} \in F$$

The structure $\mathcal{B}$ defined this way is called a reduced product, and usually denoted by $\prod_{i\in I} \mathcal{A}_i/F$.

Reduced products and powers: examples

Example (almost everywhere)

Let $\mathbf{N} = \langle \mathbb{N}, <, S \rangle$, where S is a unary relation such that $x \in S$ iff $13 < x$. Let F be the set of all cofinite subsets of $\mathbb{N}$. Then F is a filter over $\mathbb{N}$. Consider $\mathcal{B} = \mathbf{N}^{\mathbb{N}}/F$. Suppose $a/F \in S^{\mathcal{B}}$. What can we say about a?

By definition, $a/F \in S^{\mathcal{B}}$ iff $\{n \in \mathbb{N} \colon a(n) \in S^{\mathbf{N}}\} \in F$ iff $\{n \in \mathbb{N} \colon a(n) \in S^{\mathbf{N}}\}$ is cofinite iff $a(n) > 13$ for all but finitely many n.

Example (negation)

By definition, $a/F \notin S^{B}$ iff $\{i \in I \colon a(i) \in S\} \notin F$. Is it the same as $\{i \in I \colon a(i) \notin S\} \in F$?

No it is not. Consider $a(i) = 13$ if i is even and $= 14$ if i is odd.

Ultraproducts: Łoś Theorem

Let $(\mathcal{A}_i)_{i\in I}$ be a family of similar structures. Let U be an ultrafilter on $\mathcal{P}(I)$. The reduced product $\prod_{i\in I}\mathcal{A}_i/U$ is called an ultraproduct of $(\mathcal{A}_i)_{i\in I}$ (over U).

Theorem

Let $(\mathcal{A}_i)_{i\in I}$ be a family of similar structures, and U an ultrafilter over I. Let $\varphi(x_1,\ldots,x_n)$ be a first-order formula with free variables $x_1,\ldots,x_n$. Then, for any tuple $(a_0,\ldots,a_n)$ of elements of $\prod_{i\in I}\mathcal{A}_i$, the following are equivalent:

1. $\prod_{i\in I}\mathcal{A}_i/U \models \varphi[a_0/U,\ldots,a_n/U]$,
2. $\{i\in I\colon \mathcal{A}_i\models\varphi[a_0(i),\ldots,a_n(i)]\}\in U$.

Notation:

- $\mathcal{A}\models\varphi[a_0,\ldots,a_n]$ means $\mathcal{A}\models_\nu\varphi(x_0,\ldots,x_n)$, with $\nu(x_i)=a_i$.
- An element $a\in\prod_{i\in I}A_i/U$ is an equivalence class of $\sim$ for some sequence $(a_i\colon i\in I)$. So, $a(i)$ is the i-th coordinate of an arbitrary representative of that sequence.

Digression: ultraproducts for MAT4GA crowd

Let $(\mathcal{A}_i)_{i\in I}$ be a family of similar structures, and U an ultrafilter over I.

- The equaliser of elements $a, b \in \prod_{i\in I} A_i$, written $[[a = b]]$ is the set $\{i \in I \colon a(i) = b(i)\}$.
- The relation $\sim$ on $a, b \in \prod_{i\in I} A_i$ can then be defined by $a \sim b$ iff $[[a = b]] \in U$.
- The equaliser notation exends naturally to terms. For terms $t(x, y)$ and $s(x)$ and $a, b \in \prod_{i\in I} A_i$ we have $[[t(a, b) = s(a)]] = \{i \in I \colon t(a, b)(i) = s(a)(i)\}$.
- It extends further to arbitrary formulae. Let $\varphi(x, y)$ be a formula. We have $[[\varphi(a, b)]] = \{i \in I \colon \mathcal{A}_i \models \varphi(a(i), b(i))\}$.
- The equivalence in Łoś Theorem, can be then written as:

$$\prod_{i\in I}\mathcal{A}_i/U \models \varphi(a_0/U, \dots, a_n/U) \text{ iff } [[\varphi(a_0, \dots, a_n)]] \in U$$

Applications: compactness

Theorem (Compactness yet again)

Let T be a theory. Suppose every finite $T' \subseteq T$ has a model. Then, T has a model.

Proof sketch.

- Let I be the set of all finite subsets of T and for each $i \in I$ let $\mathcal{A}_i$ be a model of i.
- For each $i \in I$, define $J_i = \{j \in I : i \subseteq j\}$. Observe that $J_{i_1} \cap J_{i_2} = J_{i_1 \cup i_2}$, so the family $\mathcal{J} = (J_i)_{i \in I}$ has the finite intersection property.
- Take an ultrafilter U over I extending $\mathcal{J}$, and consider $\prod_{i \in I} \mathcal{A}_i / U$.
- For each $\varphi \in T$, we have $\{\varphi\} \in I$ (say, $\{\varphi\} = i_0$), so $J_{i_0} \in \mathcal{J}$.
- But $J_{i_0} = \{i \in I : \mathcal{A}_i \models \varphi\}$, so apply Łoś.

□

Embedding into finitely generated substructures

Theorem

Every structure is embeddable in an ultraproduct of its finitely generated substructures.

Proof sketch.

- Let I be the set of all finite subsets of A, and for each $i \in I$ let $\mathcal{A}_i$ be substructure of $\mathcal{A}$ generated by i.
- For each $i \in I$, define $J_i = \{j \in I : i \subseteq j\}$. Observe that $J_{i_1} \cap J_{i_2} = J_{i_1 \cup i_2}$, so the family $\mathcal{J} = (J_i)_{i \in I}$ has the finite intersection property.
- Take an ultrafilter U over I extending $\mathcal{J}$, and consider $\prod_{i \in I} \mathcal{A}_i / U$.
- Take the map $\mu : A \to \prod_{i \in I} A_i$, defined by $\mu(a) = (a_i : i \in I)$ where $a_i = a$ if $a \in A_i$ and is arbitrary otherwise. This is the embedding we want.

□

Elementary embedding into ultrapower

Let $\mathcal{A}$ be a structure, $\mathcal{A}^I/U$ an ultrapower of $\mathcal{A}$, and a an element of A. We write a/U for the element $\langle b(i) \colon i \in I \rangle / U$ where $b(i) = a$ for every $i \in I$.

Lemma

The map $a \mapsto a/U$ is an elementary embedding of $\mathcal{A}$ into $\mathcal{A}^I/U$.

Proof sketch.

- It is clearly an embedding.
- We use Tarski-Vaught test. Let $\varphi(x)$ be a formula with only x free (forget about parameters, for simplicity).
- Suppose $\mathcal{A}^I/U \models \exists x \colon \varphi(x)$. Then, for some $a/U = (a_i \colon i \in I)/U$, we have $\mathcal{A}^I/U \models \varphi[a]$.
- Moreover, by Łoś, $\{i \in I \colon \mathcal{A}_i \models \varphi[a_i]\} \in U$.
- Pick any i in the set above. $\mathcal{A}_i = \mathcal{A}$, so we get $\mathcal{A} \models \varphi[a_i]$ as needed.

$\square$

Fancy models via ultraproducts

Example (From little things...)

Let $\mathcal{A}_n$ be the n-element chain (as an ordered set). Let U be a non-principal filter over ω. Then, $\prod_{i \in I} \mathcal{A}_i / U$ is an infinite chain consisting of a copy of ω at the bottom, a copy of dual ω at the top, and uncountably many copies of $\mathbb{Z}$ in between.

Example (Non-standard natural numbers)

Let U be a non-principal ultrafilter over ω. Consider $\mathbb{N}^\omega / U$. It is elementarily equivalent to $\mathbb{N}$. Take the element $(1, 2, 3, 4, \dots)/U$. It is strictly greater than any standard natural number.

Example (Non-standard reals)

Let U be a non-principal ultrafilter over ω. Consider $\mathbb{R}^\omega / U$. It is elementarily equivalent to $\mathbb{R}$. Take the element $(1, 1/2, 1/3, 1/4, \dots)/U$. It is strictly greater than 0, yet strictly smaller than any standard real number.

Non-elementarity via ultraproducts

A class $\mathbb{C}$ of structures is called elementary, if $\mathbb{C}$ is the class of models of some set T of formulae (notation: $\mathbb{C} = \mathrm{Mod}(T)$).

Lemma

Let $\mathbb{C}$ be the class of finite structures of some signature τ. If $\mathbb{C}$ contains arbitrarily large finite structures, then $\mathbb{C}$ is not elementary.

Proof.

Suppose $\mathbb{C} = \mathrm{Mod}(T)$ for some theory T. Let $(\mathcal{A}_i : i \in \omega)$ be a sequence of structures from $\mathbb{C}$ with strictly increasing sizes of their universes. Take $\mathcal{A} = \prod_{i \in \omega} \mathcal{A}_i / U$. Then, $\mathcal{A} \models T$, but $\mathcal{A}$ is infinite, so $\mathbb{C} \subset \mathrm{Mod}(T)$, contradicting the assumption. □

Theorem

The following classes are not elementary:

1. *the class of all torsion groups.*
2. *the class of all connected graphs.*
3. *the class of all fields of non-zero characteristic.*

Example: pseudo-finite fields

A field $\mathbf{F}$ is pseudo-finite if it satisfies all first-order properties of all finite fields.

Example (Pseudo-finite fields)

Let $(\mathbf{F}_i)_{i\in\omega}$ be an enumeration of all finite fields. Let U be a non-principal ultrafilter over ω. Then, $\mathbf{F} = \prod_{i\in\omega} \mathbf{F}_i/U$ is an infinite pseudo-finite field.

Question

What is the characteristic of $\mathbf{F}$*? That is, do we have* $\prod_{i\in\omega} \mathbf{F}_i/U \models \underbrace{1+\cdots+1}_{p \text{ times}} = 0$*, for any prime p?*

That depends on the choice of U.

Pseudo-finite fields continued

Consider the following enumeration of all finite fields as a $\omega \times \omega$ matrix.

$$\begin{pmatrix} GF(2) & GF(3) & \dots & GF(p) & \dots \\ GF(2^2) & GF(3^2) & \dots & GF(p^2) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

- Choose U so that it "contains a column". Then, $\prod_{i\in\omega} \mathbf{F}_i/U$ is a pseudo-finite field of characteristic p.
- Let $S_p \subset \omega \times \omega$ be a set "omitting a column". The collection $(S_p)_{p \text{ prime}}$ has finite intersection property, so it extends to an ultrafilter U. In this case, $\prod_{i\in\omega} \mathbf{F}_i/U$ is a pseudo-finite field of characteristic 0.
- Observe that in neither case is $\prod_{i\in\omega} \mathbf{F}_i/U$ algebraically closed. To be algebraically closed it would have to satisfy the sentence

$$\forall y_0, \dots, y_n \; \exists x \colon y_n x^n + y_{n-1} x^{n-1} + \dots + y_1 x + y_0 = 0$$

for every n. But it does not.

Quantifier elimination and model completeness

Quantifier elimination

Definition

Let L be a signature an $\mathbb{K}$ a class of L-structures. A set Φ of formulae is called an elimination set for $\mathbb{K}$, if for every formula $\varphi(\overline{x})$ of L there is a formula $\varphi^*(\overline{x})$ such that

- $\varphi^*(\overline{x})$ is a Boolean combination of formulae in Φ,
- For every $\mathcal{A} \in \mathbb{K}$ and every tuple $\overline{a}$ from A we have $\mathcal{A} \models \varphi(\overline{a})$ iff $\mathcal{A} \models \varphi^*(\overline{a})$.

If there is an elimination set Φ for $\mathbb{K}$, such that all formulae in Φ are quantifier free, then we say that $\mathbb{K}$ admits quantifier elimination. If $\mathbb{K} = \mathrm{Mod}(T)$ for some theory T, we also say that T admits quantifier elimination.

Lemma

If every formula $\exists x\colon \alpha_1 \wedge \cdots \wedge \alpha_n$ where each α_i is atomic or negated atomic, is equivalent over T to a quantifier free formula, then T admits quantifier elimination.

Quantifier elimination: dense linear orders

Example

Let $\mathbb{K}$ be the class of dense linear orders. Define Φ to be

- $\exists x \forall y \colon x = y \lor x < y$,
- $\exists x \forall y \colon x = y \lor y < x$,
- $\forall y \colon x = y \lor x < y$,
- $\forall y \colon x = y \lor y < x$,
- $x < y$.

Then Φ is an elimination set for $\mathbb{K}$.

Example

Let T be the following theory:

- Axioms of dense strict linear order.
- $\forall x \exists y \colon x < y$.
- $\forall x \exists y \colon y < x$.

Then T admits quantifier elimination.

Quantifier elimination and decidability

The following classes/theories admit quantifier elimination:

- Dense linear orders without endpoints.
- Presburger arithmetic with divisibility predicates: it is the theory of $(\mathbb{Z}; <, (D_k)_{k\in\mathbb{N}}, +, -, 0, 1)$, where $D_k(x)$ holds if x is divisible by k.
- Algebraically closed fields: fields satisfying $\forall x_1 \dots x_n \exists y\colon y^n + x_1 y^{n-1} + \cdots + x_{n-1} y + x_n = 0$, for every $n \in \mathbb{N}$
- Real closed fields: fields satisfying
  $\forall x_1 \dots x_n\colon x_1^2 + \cdots + x_n^2 \neq -1$, for every $n \in \mathbb{N}$
  $\forall x \exists y\colon x = y^2 \vee -x = y^2$
  $\forall x_1 \dots x_n \exists y\colon y^n + x_1 y^{n-1} + \cdots + x_{n-1} y + x_n = 0$, for every odd n.

The main early use of quantifier elimination was to establish decidability of a theory T.

Model completeness

Definition

A theory T is model complete if every embedding between models of T is elementary. That is, for every $\mathcal{A}, \mathcal{B} \in \mathrm{Mod}(T)$ we have $\mathcal{A} \leq \mathcal{B}$ if and only if $\mathcal{A} \prec \mathcal{B}$.

Theorem

Let T be a theory. If T admits quantifier elimination, then T is model complete.

Proof.

Let $\mathcal{A}$, $\mathcal{B}$ be models of T such that $\mathcal{A} \leq \mathcal{B}$. To use Tarski-Vaught test, take an arbitrary existential formula $\exists x \colon \varphi(x, \overline{a})$ with parameters from A. Suppose $\mathcal{B} \models \exists x \colon \varphi(x, \overline{a})$. By quantifier elimination, there is a quantifier free formula $\varphi^*(x, \overline{y})$, equivalent to $\varphi(x, \overline{y})$ over all models of T (and all tuples of parameters). Thus, $\mathcal{B} \models \varphi^*(x, \overline{a})$. Since φ^* is quantifier free, it is equivalent to a universal formula; hence $\mathcal{A} \models \varphi^*(x, \overline{a})$ by Łoś-Tarski preservation theorem. By quantifier elimination again $\mathcal{A} \models \exists x \colon \varphi(x, \overline{a})$. □

A version of Hilbert's Nullstellensatz

Theorem (Hilbert)

Let $\mathcal{F}$ be an algebraically closed field, and E a finite system of equations and inequations over $\mathcal{F}$ in variables $\overline{x}$. Suppose E has a solution in some field $\mathcal{G}$ extending $\mathcal{F}$. Then, E has a solution in $\mathcal{F}$.

Proof.

That $\mathcal{G}$ extends $\mathcal{F}$, means $\mathcal{F} \leq \mathcal{G}$. Let $\overline{\mathcal{G}}$ be the algebraic closure of $\mathcal{G}$. (It exists: this is by algebra). Then, we have $\mathcal{F} \leq \mathcal{G} \leq \overline{\mathcal{G}}$. Now, $E = \{p_1(\overline{x}) = 0, \ldots, p_n(\overline{x}) = 0, q_1(\overline{x}) \neq 0, \ldots, q_m(\overline{x}) \neq 0\}$ for some polynomials $p_1, \ldots, p_n, q_1, \ldots, q_m$ with coefficients from $\mathcal{F}$. Let $\psi(\overline{x}, \overline{c})$ be the conjunction of all formulae in E, with $\overline{c}$ being the coefficients. Then, E has a solution in $\mathcal{G}$ means $\mathcal{G} \models \exists \overline{x} \colon \psi(\overline{x}, \overline{c})$. This is an existential formula, so by Łoś-Tarski $\overline{\mathcal{G}} \models \exists \overline{x} \colon \psi(\overline{x}, \overline{c})$. Since the theory of algebraically closed fields is model complete, the embedding $\mathcal{F} \leq \overline{\mathcal{G}}$ is elementary, that is, $\mathcal{F} \prec \overline{\mathcal{G}}$. So, $\mathcal{F} \models \exists \overline{x} \colon \psi(\overline{x}, \overline{c})$ proving the claim. □

Ehrenfeucht-Fraïssé games: isomorphism, elementary equivalence and things in between

Elementary equivalence without games

Theorem (Keisler and Shelah)

Two similar structures A and B are elementarily equivalent if and only if there is a set I and an ultrafilter $\mathcal{U}$ on I such that $A^I/\mathcal{U} \cong B^I/\mathcal{U}$.

An easier theorem is the following crucial fact.

Theorem

$A \equiv B$ if and only if there is an elementary embedding of A into an ultrapower of B. More generally, the following is true for any class K of structures of the same signature L:

1. *K is the class of all L-models of some set of first order sentences;*
2. *K is closed under isomorphisms, elementary embeddings and ultraproducts;*
3. *$K = IEP_U(K')$ for some class of L-structures K'.*

A test for strict elementarity

A class K in some signature L is called strictly elementary, if K is the class of models of a single first order sentence.

Corollary

If K is strictly elementary, then the class of L-models not in K is closed under taking ultraproducts.

Examples.

- Let K be a class containing arbitrarily large finite structures. The class K_∞ of all infinite structures in K is not strictly elementary. It is elementary, if K is.
- The class of all models of the theory of the random graph.
- The class of Representable Relation Algebras is not finitely axiomatisable by identities. (Tarski)

Isomorphism, and elementary equivalence compared

When are two models are "the same"? It depends. "The same" could mean, for example:

- Isomorphic, that is, indistinguishable by anything at all.
- Elementarily equivalent, that is, indistinguishable by any first-order sentences.

A related question: can a theory T have only one model, up to isomorphism?

- Yes, if T has no infinite models.
- No, by upward Löwenheim-Skolem, if T has infinite models.

Definition

A theory T is ω-categorical if every two countable models of T are isomorphic.

Ehrenfeucht-Fraïssé games

Elementary equivalence by games.

A method for determining elementary equivalence; due to Fraïssé (1950), and independently A. D. Taimanov (1950s?), and formulated in game theoretic terms by Ehrenfeucht (1961).

Two player game with perfect information

$\forall$	$\exists$
Spoiler	Duplicator
Samson	Delilah
Abelard	Eloise (Heloïse)

Back and forth games: fix similar structures A and B.

If $\exists$ has won the first $j \geq 0$ rounds then there are sequences $(a_1, \ldots, a_j) \in A$ and $(b_1, \ldots, b_j) \in B$ such that $\langle a_1, \ldots, a_j \rangle_A \leq A$ is isomorphic $\langle b_1, \ldots, b_j \rangle_B \leq B$ under the map $a_i \mapsto b_i$. In particular, $\exists$ wins the first 0 rounds if the substructures of A and B generated by the constants are isomorphic.

Stage $j + 1$.

Part 1. $\forall$ selects some $a_{j+1} \in A$ (or $b_{j+1} \in B$).

Part 2. $\exists$ must select a $b_{j+1} \in B$ (or $a_{j+1} \in A$, respectively) such that $\langle a_1, \ldots, a_{j+1} \rangle$ is isomorphic to $\langle b_1, \ldots, b_{j+1} \rangle$ under the map $a_i \mapsto b_i$.

$\exists$ wins round $j + 1$ is she can complete part 2. Otherwise $\forall$ wins and the game does not continue into further rounds.

Note: if $\forall$ selects from A, then $\exists$ must select from B, and dually.
Note: don't forget constants in the signature and possible equalities within $\{a_1, \ldots, a_{j+1}\}$ and $\{b_1, \ldots, b_{j+1}\}$

Back and forth equivalence

Definition

Write $A \sim_\omega B$ if $\exists$ has a winning strategy for the ω play of the back and forth game.

Theorem (Isomorphism by back-and-forth game)

For countable structures A, B, we have $A \cong B$ if and only if $A \sim_\omega B$.

Proof sketch.

($\Rightarrow$) An isomorphism $\iota : A \to B$ gives $\exists$ an obvious winning strategy.
($\Leftarrow$) If $A \sim_\omega B$, list the elements of A and B, and let $\forall$ choose alternately the first unassigned element of A and B. Since A and B are countable, this produces a bijection ι between A and B. It remains to check that ι is an isomorphism. Suppose it is not. Then, there is some relation R and elements $a_1, \ldots, a_k$ of A such that $(a_1, \ldots, a_k) \in R^A$ but $(\iota(a_1), \ldots, \iota(a_k)) \notin R^B$ (or the other way round). But, the elements $a_1, \ldots, a_k$ and $\iota(a_1), \ldots, \iota(a_k)$ have been played at some finite stage of the game, so $\exists$ would have lost then. Contradiction. $\square$

Observations and examples

- For uncountable A and B, the relation $\sim_\omega$ is too weak to determine isomorphism, and may be too strong to determine elementary equivalence.
- Back and forth equivalence (that is, $\sim_\omega$) is an equivalence relation.

Examples

1. The theory of the random graph is ω-categorical (it is homogeneous).
2. The theory of dense linear orders with end points is ω-categorical (but is not homogeneous).
3. The theory of “discrete chain without top or bottom” is *not* ω-categorical (but is a complete theory).
4. Two algebraically closed fields of the same characteristic and with infinite transcendence degree are back and forth equivalent.

Finite games on *relational* structures.

Definition (Quantifier rank)

- Atomic formulas Φ have quantifier rank $\mathrm{qr}(\Phi) = 0$,
- $\mathrm{qr}(\Phi_1 \wedge \Phi_2) = \mathrm{qr}(\Phi_1 \vee \Phi_2) := \max\{\mathrm{qr}(\Phi_1), \mathrm{qr}(\Phi_2)\}$,
- $\mathrm{qr}(\neg\Phi) := \mathrm{qr}(\Phi)$,
- $\mathrm{qr}(\forall x \Phi(x, \dots)) = \mathrm{qr}(\exists x \Phi(x, \dots)) := \mathrm{qr}(\Phi(x, \dots)) + 1$.

Note that logically equivalent formulas may have different quantifier ranks: take $\exists x \forall y \colon x > 0 \wedge 1 > y$ and $\exists x \colon x > 0 \wedge \forall y \colon 1 > y$.

Definition

For structures A, B write $A \equiv_k B$ if for every sentence Φ of quantifier rank at most k we have $A \models \Phi$ if and only if $B \models \Phi$.

Note that $A \equiv B$ is the same as $A \equiv_k B$ for all $k \in \omega$.

Types

Definition

An *n-type* of a theory T is a set of formulas $\Phi(\bar{x})$, in variables $\bar{x} = x_1, \ldots, x_n$, such that for some $A \models T$ and some tuple $\bar{a}$ from A, we have $A \models \Phi(\bar{a})$.

If things are as the definition says, we say that

- A realises Φ, or $\bar{a}$ realises Φ in A.

If no tuple from A realises Φ, we say that A omits Φ.

The analysis of types is a large part of model theory.

Definition

Let A be a structure and $\bar{a} \in A^n$ for some n. The *n-type* $\mathrm{tp}(A, \bar{a})$ *of $\bar{a}$ in A* is the set of formulas $\{\phi(\bar{x}) \colon A \models \phi(\bar{a})\}$. When n is clear this is just called the *type* of $\bar{a}$ in A.

- The *rank-k* type of $\bar{a}$ in A is the restriction of $\mathrm{tp}(A, \bar{a})$ to formulas of rank k. It can be denoted $\mathrm{tp}_k(A, \bar{a})$.

Game equivalence

We will consider A, B of the same, finite, relational signature, possibly with finitely many constants. For a full version, see SMT, Sec. 3.3. Thm. 3.3.2.

Definition (Revisiting definition of $\sim_\omega$)

- Write $A \sim_k B$ if $\exists$ has a strategy for winning the k-round back and forth game.
- Observe that $\exists$ can have a strategy for winning the k-round game for every k, but not a strategy for winning the infinite back and forth game.

Theorem (Game equivalence)

1. *$A \equiv_k B$ if and only if $A \sim_k B$.*
2. *$A \equiv B$ if and only if $A \sim_k B$ for every $k \in \omega$.*
3. *If A, B are countable (including finite), then $A \cong B$ if and only if $A \sim_\omega B$ (this is isomorphism by games theorem).*

Proof of game equivalence theorem: a lemma.

Part 2 of the theorem follows from part 1 and part 3 is already proved, so we focus on part 1.

Lemma

1. *Up to logical equivalence, there are only finitely many rank-k formulas in m free variables $x_1, \ldots, x_m$.*
2. *There are only finitely many rank-k 1-types in the language L; say $T_1(x), \ldots, T_n(x)$.*
3. *For each rank-k 1-type $T_i(x)$, there is a rank-k formula $\alpha_i(x)$ such that $\forall x \colon T_i(x) \leftrightarrow \alpha_i(x)$.*

Proof. (1) By induction on k. It is trivial for $k = 0$ as the signature is finite and relational. For $k + 1$ (assuming truth at k), note that each formula of rank $k + 1$ is a Boolean combination of formulas of the form $\exists x_{m+1} \colon \varphi(x_1, \ldots, x_m, x_{m+1})$, where $\phi(x_1, \ldots, x_{m+1})$ is of rank k; there are (by hypothesis) only finitely many such formulas.
(2) follows from (1) and (3) follows from (2) (take conjunction). $\square$

Preparing for a proof of game equivalence theorem

Lemma (Important observation)

$(A, a) \equiv_k (B, b)$ *if and only if* $\mathrm{tp}_k(A, a) = \mathrm{tp}_k(B, b)$.

Proof.

- $(A, a) \equiv_k (B, b)$ holds iff for every rank-k sentence φ, in the language expended by a single constant c with $c^A = a$ and $c^B = b$, we have $(A, a) \models \varphi$ iff $(B, b) \models \varphi$.
- $\mathrm{tp}_k(A, a) = \mathrm{tp}_k(B, b)$ holds iff for every rank-k formula $\psi(x)$, with at most one free variable, we have $A \models \psi[a]$ iff $B \models \psi[b]$ (using square brackets to indicate that x is evaluated by a and b, respectively).
- Since $A \models \psi[a]$ iff $(A, a) \models \psi(c^A)$, and similarly for B, we get that the two statements above are the same. □

Lemma (Obvious fact)

- $A \equiv_k B$ *implies* $A \equiv_\ell B$ *for every* $\ell \leq k$.

Proof of (1. $\Leftarrow$): induction on k

- Base case: $A \sim_0 B$ means that Eloise has a winning strategy for a 0-round game. $A \equiv_0 B$ means that substructures of A and B consisting of constants only are isomorphic. These statements are clearly equivalent.
- Assume $A \sim_{k+1} B$, and consider a sentence σ of rank $k+1$.
- Using $\neg\forall\neg = \exists$, we can write σ as a boolean combination of sentences $\exists x \colon \varphi(x)$, with $\varphi(x)$ a formula of rank (at most) k.
- It suffices to show that for every such sentence ψ we have $A \models \psi$ implies $B \models \psi$ (converse will follow by symmetry).
- So, suppose $A \models \exists x \colon \varphi(x)$. Take a witness $a \in A$.
- Let Abelard play a as a first move of a game on A and B. Eloise responds by $b \in B$.
- Now, we play a k round game on (A, a) and (B, b). By assumption, we have $(A, a) \sim_k (B, b)$.
- By inductive hypothesis, $(A, a) \equiv_k (B, b)$. Since $\varphi(a)$ is a sentence rank k, and $(A, a) \models \varphi(a)$, we get $B \models \varphi(b)$.

Proof of (1. $\Rightarrow$): induction on k

- Assume $A \equiv_{k+1} B$, and consider the first move of play in the a $(k+1)$-round back and forth game.
- Suppose Abelard selects $a \in A$. Consider the rank-k 1-type $\mathrm{tp}_k(A, a)$ of a. Say, $\mathrm{tp}_k(A, a) = T(x)$.
- By the "important observation" lemma, there is a single formula $\alpha(x)$ of rank-k such that $\forall x\colon T(x) \leftrightarrow \alpha(x)$ holds.
- Thus, $A \models \alpha(a)$ iff $A \models T(a)$, and $B \models \alpha(b)$ iff $B \models T(b)$.
- By definition of $\mathrm{tp}_k(A, a)$, we have $A \models T(a)$. So, $A \models \exists x\colon \alpha(x)$; a sentence of rank $k+1$.
- As $A \equiv_{k+1} B$, we have that $B \models \exists x\colon \alpha(x)$; thus $B \models \alpha(b)$ for some $b \in B$, and so $B \models T(b)$.
- So $(A, a) \equiv_k (B, b)$. Then by induction hypothesis, we have $(A, a) \sim_k (B, b)$.
- It follows by induction that Eloise may survive a further k rounds. Thus $A \sim_{k+1} B$ as required.

HP, JEP and AP

Definition

A class $\mathbb{K}$ of structures some of signature L has hereditary property (HP), if whenever $\mathcal{B} \in \mathbb{K}$ and $\mathcal{A} \leq \mathcal{B}$, then $\mathcal{A} \in \mathbb{K}$.

Definition

A class $\mathbb{K}$ of structures some of signature L has joint embedding property (JEP), if for every $\mathcal{A}, \mathcal{B} \in \mathbb{K}$ there exists a $\mathcal{C} \in \mathbb{K}$ and embeddings $f\colon \mathcal{A} \to \mathcal{C}$ and $g\colon \mathcal{B} \to \mathcal{C}$.

Example

Let $\mathcal{K}$ be a single structure, and $\mathbb{K}$ the set of all finitely generated substructures of $\mathcal{K}$. Then, $\mathcal{K}$ has HP and JEP.

Definition

A class $\mathbb{K}$ of structures some of signature L has amalgamation property (AP), if for every $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{K}$, and embeddings $f\colon \mathcal{C} \to \mathcal{A}$ and $g\colon \mathcal{C} \to \mathcal{B}$, there exists a $\mathcal{D} \in \mathbb{K}$ and embeddings $h\colon \mathcal{A} \to \mathcal{D}$ and $k\colon \mathcal{B} \to \mathcal{D}$, such that for every $c \in C$ we have $h(f(c)) = k(g(c))$.

Age and homogeneity

Definition

Let $\mathcal{K}$ be any structure. The age of $\mathcal{K}$, written $\mathrm{age}(\mathcal{K})$, is the set of all finitely generated substructures of $\mathcal{K}$.

Example

Consider $\mathbb{Q}$ and $\mathbb{Z}$ as ordered sets. Then,
$\mathrm{age}(\mathbb{Q}) = \mathrm{age}(\mathbb{Z}) = \{$finite linearly ordered sets$\}$.

Definition

A structure $\mathcal{K}$ is homogeneous if any isomorphism $f : \mathcal{A} \to \mathcal{B}$ between finitely generated substructures $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{K}$ extends to an automorphism of $\mathcal{K}$.

Example

Consider $\mathbb{Q}$ and $\mathbb{Z}$ as ordered sets again. Clearly $\mathbb{Z}$ is not homogeneous: consider $0 \mapsto 0$, $2 \mapsto 1$. But $\mathbb{Q}$ is homogeneous. In this sense, $\mathbb{Q}$ is a better "limit" of its age, than $\mathbb{Z}$.

Anything that can be an age, is

Theorem

Let L be a signature and let $\mathbb{K}$ be a nonempty finite or countable set of finitely generated L-structures which has HP and JEP. Then, $\mathbb{K}$ is the age of some finite or countable structure.

Proof.

Enumerate $\mathbb{K}$ as $(A_i \colon i \in \omega)$, possibly with repetitions. Put $B_0 = A_0$. Using JEP, construct a chain of embeddings

$$\begin{array}{ccccccc} A_0 = B_0 & \rightarrowtail & B_1 & \rightarrowtail & B_2 & \rightarrowtail & B_3 \\ & \nearrow & & \nearrow & & \nearrow & \\ & A_1 & & A_2 & & A_3 & \end{array}$$

Define $B = \bigcup_{i \in \omega} B_i$. By construction and HP, every $A_i \in \mathbb{K}$, so $\mathbb{K} \subseteq \mathrm{age}(B)$. Next, if $C \leq B$ is finitely generated, say by X, then $X \subseteq B_i$ for some i, so $C \in \mathbb{K}$, showing that $\mathrm{age}(B) \subseteq \mathbb{K}$. □

Fraïssé Theorem

Theorem (Fraïssé)

Let L be a countable signature and let $\mathbb{K}$ be a nonempty finite or countable set of finitely generated L-structures which has HP, JEP and AP. Then, there is an L-structure $\mathcal{D}$ (called Fraïssé limit of $\mathbb{K}$), unique up to isomorphism, such that

1. *$\mathcal{D}$ is at most countable,*
2. $\mathbb{K} = \mathrm{age}(\mathcal{D})$,
3. *$\mathcal{D}$ is homogeneous.*

Definition

A structure D is weakly homogeneous if it has the following property:

- if A, B are finitely generated substructures of D, and $A \leq B$, and $f : A \to D$ is an embedding, then there is an embedding $g : B \to D$ that extends f.

Clearly, if D is homogeneous, it is weakly homogeneous.

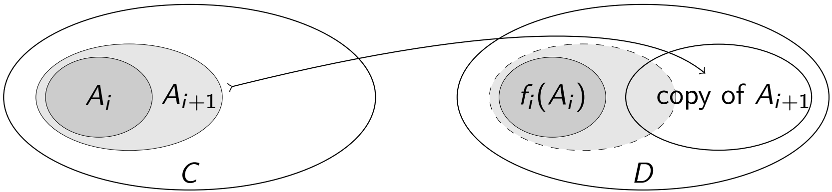
Uniqueness: weak homogeneity and embeddings

Lemma

Let C and D be L-structures which are both at most countable. Suppose $\mathrm{age}(C) \subseteq \mathrm{age}(D)$ and D is weakly homogeneous. Then, any embedding from a finitely generated substructure of C into D can be extended to an embedding of C into D; in particular, C is embeddable into D.

Proof.

Let $f_0 \colon A_0 \to D$ be an embedding of a finitely generated $A_0 \leq C$ into D. We extend it to $f_\omega \colon C \to D$ as follows. Write C as $\bigcup_{i \in \omega} A_i$ of a chain of finitely generated substructures of C.
Inductive step by picture:



Uniqueness: weak homogeneity and homogeneity

Lemma

(a) *Let C and D be L-structures with $\mathrm{age}(C) = \mathrm{age}(D)$. Suppose C and D are at both most countable and weakly homogeneous. Then, $C \cong D$; in fact, if A is a finitely generated substructure of C and $f : A \to D$ is an embedding, then f extends to an isomorphism from C to D.*

(b) *A finite or countable structure is weakly homogeneous if and only if it is homogeneous.*

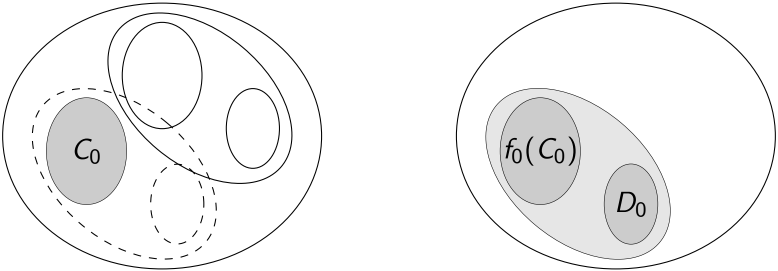
Proof.

Write $C = \bigcup_{i \in \omega} C_i$ and $D = \bigcup_{i \in \omega} D_i$, each a countable union of a chain of finitely generated substructures. We will define a chain of isomorphisms f_i between finitely generated substructures of C and D, so that, for each n

- domain of f_{2n} includes C_n,
- range of f_{2n+1} includes D_n.

Uniqueness: proof completed

Inductive step (from $n = 0$ to $n = 1$) by picture:



- Start by picking f_0: an embedding of C_0 into D.
- The light grey blob, say G, is the substructure of D generated by $f_0(C_0) \cup D_0$.
- It has an image, say $g(G)$, inside C (solid lines).
- Take the identity embedding of C_0 into C, and g. By weak homogeneity, the identity on C_0 extends to an[other] embedding, say h, of G into C (dashed lines).
- Put $f_1 = h^{-1}$.

Existence of Fraïssé limit

Lemma (easy observation)

Let $\mathbb{J}$ be a set of finitely generated L-structures, and $(D_i : i \in \omega)$ be a chain of L-structures.

- *If* $\mathrm{age}(D_i) \subseteq \mathbb{J}$ *for every* $i \in \omega$*, then* $\mathrm{age}(\bigcup_{i\in\omega} D_i) \subseteq \mathbb{J}$.
- *If* $\mathrm{age}(D_i) = \mathbb{J}$ *for every* $i \in \omega$*, then* $\mathrm{age}(\bigcup_{i\in\omega} D_i) = \mathbb{J}$.

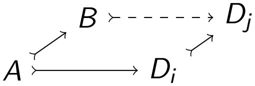
Existence proof.

Assume $\mathbb{K}$ is nonempty, has HP, JEP and AP, and (wlog) is closed under taking isomorphic copies. Assume also that $\mathbb{K}$ contains only countably many isomorphism types of structure (an assumption automatically satisfied if $\mathbb{K}$ is itself at most countable).
Suppose we have constructed a chain $(D_i : i \in \omega)$ of structures from $\mathbb{K}$ such that the following holds:

($\star$) If $A, B \in \mathbb{K}$ and $A \leq B$, and there is an embedding $f : A \to D_i$ for some $i \in \omega$, then there is an embedding $g : B \to D_j$ extending f.

Existence: proof outlined

Pictorially, $(\star)$ is:



Then, put $D = \bigcup_{i \in \omega} D_i$. By the "easy observation" lemma, we have $\mathrm{age}(D) \subseteq \mathbb{K}$. Conversely, if $A \in \mathbb{K}$, then by JEP, there is a $B \in \mathbb{K}$ such that both A and D_0 are embeddable in B. Wlog (as $\mathbb{K}$ is closed under taking isomorphic copies), $A \leq B$. Then, by $(\star)$ we have an embedding of B into some D_j. Then, we have

$$A \leq B \in \mathrm{age}(D_j) \subseteq \mathrm{age}(D)$$

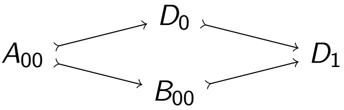
and so $A \in \mathrm{age}(D)$. Therefore $\mathrm{age}(D) = \mathbb{K}$, and so D satisfies (1) and (2) of Fraïssé Theorem. For (3) observe that D is weakly homogeneous by construction, but in the uniqueness part we have established that weak homogeneity is the same as homogeneity. All that remains now is to construct the chain $(D_i \colon i \in \omega)$.

Constructing the chain

Let P be a countable set containing a representative of each isomorphism type of a pair (A, B) such that $A \leq B \in \mathbb{K}$. Fix some enumeration of $\omega \times \omega$, for example

	0	1	2	3	...
0	0	1	4	9	
1	3	2	5	10	
2	8	7	6	11	
3	15	14	13	12	

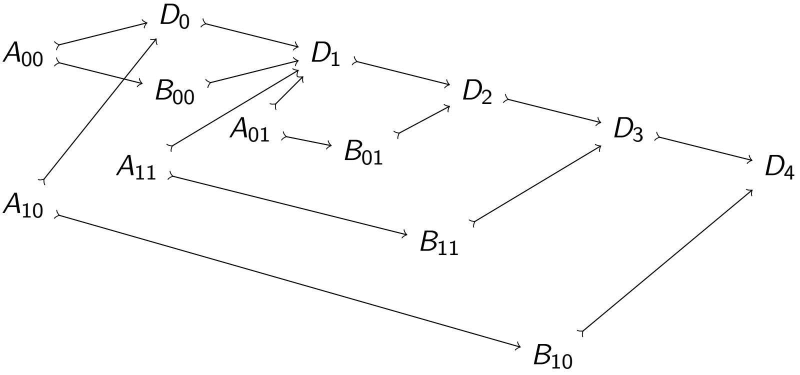
Pick D_0 from $\mathbb{K}$ arbitrarily. Enumerate all pairs (A, B) from P, with A embeddable in D_0, as (f_{i0}, A_{i0}, B_{i0}), where $f_{i0}\colon A_{i0} \to D_0$ is the embedding (this fills the first column of the matrix above). Consider $f_{00}\colon A_{00} \to D_0$. By AP, we can pick a $D_1 \in \mathbb{K}$ that makes the diagram



commute.

Constructing the chain

Now that D_1 has been picked, we enumerate as (f_{i1}, A_{i1}, B_{i1}) all pairs (A, B) from P, with A embeddable in D_1. The first entry that has not yet been considered is $f_{01}\colon A_{01} \to B_{01}$. Use AP again to obtain D_2. Continuing inductively we arrive at the following picture



Constructing the chain

Now, observe:

- All pairs (A, B) with $A, B \in \mathbb{K}$ eventually appear in the enumeration.
- By construction of the chain $(D_i \colon i \in \omega)$, we can strengthen the previous statement to: all pairs (A, B) with $A, B \in \mathbb{K}$ and with A embeddable in some D_i appear in the enumeration.
- By construction of the chain again, for each such pair (A, B) and embedding $f \colon A \to D_i$, we can extend f to an embedding $g \colon B \to D_j$, for some D_j appearing "later" than D_i.

Thus, $(D_i \colon i \in \omega)$ indeed satisfies $(\star)$. This completes the proof of Fraïssé Theorem. □

Examples of Fraïssé classes

- class of finite chains (with strict order).
  - Fraïssé limit is $\langle \mathbb{Q}, < \rangle$
- class of finite graphs
  - Fraïssé limit is the Random Graph
- class of finite groups
  - Fraïssé limit is the random finite group

The Random Graph (Rado graph)

Fix $0 < p < 1$

On ω: include edge $\{i, j\}$ (for $i \neq j$) with probability p.

Enumerate all ordered pairs $(A_1, B_1), (A_2, B_2), \ldots$ of finite subsets of ω

- Let G_0 be the graph on ω with no edges.
- Assume G_k has been constructed satisfying: for each $i \leq k$ there is a vertex n_i adjacent to all vertices in A_i and none in B_i.
- To construct G_{k+1}: let $n \in \omega$ be outside of $\bigcup_{0 \leq i \leq k} A_i \cup \bigcup_{0 \leq i \leq k} B_i$. Add all edges from elements of A_{k+1} to n (and none others).

0-1 limit laws for finite graphs

A first order sentence ϕ in the language of graphs is said to hold *almost surely* if the proportion ϕ_n of graphs on n vertices satisfying ϕ approaches 1 as $n \to \infty$.

Theorem

1. The first order theory of the random graph is axiomatised by the sentences asserting "for every pair of disjoint finite sets A and B there is a vertex adjacent to all vertices in A and none in B".
2. These sentences hold almost surely on finite graphs.
3. for every first order sentence ϕ, either ϕ or $\neg\phi$ holds almost surely.